THREE QUATERNION PAPERS 2006-2007

Peter Michael Jack

Timefoot Books

“And how the One of Time,
of Space the Three,
Might in the Chain of Symbols girdled be.”

— William Rowan Hamilton 1805-1865

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Bio: P.M. Jack (b.1961-) received his Bachelors of Arts in Physics from Columbia College in 1984 and Masters of Arts in Physics from the Graduate School of Arts and Sciences in 1986, at Columbia University in New York. He has worked as a Financial Analyst and C Programmer developing and implementing mathematical algorithms for pricing and hedging derivative securities, and maintains his interest in Mathematical Physics through his part time investigation and research into the quaternions of Hamilton. These papers represent some of his important original research, which culminate in the solution to a previously unsolved quaternion problem, with the presentation of a unique method for solving linear quaternion problems that appears nowhere else in mathematical literature; hence, this text has been compiled to record these novel results.

Included Papers:

- Hexpentaquaternions: A two-hand Quaternion Algebra, January 26, 2006. hypcx-20060129a
- Quatro-Quaternions and the matrix representations of octonions. July 2, 2006. hypcx-20060630a
- General solutions to linear problems in quaternion variables. November 29, 2007. hypcx-20071128a

INTRODUCTION

VECTOR: It is a strange thing, that time and space should have one and three, and there be found a symbolic model like the quaternion to parallel this structure. The correspondences are astonishing, to say the least. Space is handed, meaning that one can turn about, rotate, or spin, within the three degrees, and a plane mirror turns left hand into right hand. So, too, do the quaternion's imaginary part, with its (i.j.k) elements, capture this feature. And yet, a 3-space constructed with one real and two imaginary axes, like (1,i,j) cannot reflect this handed nature. " Time " is not "orientable" with reference to the spatial dimensions. Symbolically, the quaternion says, 1.j = +j.1 and [1,j] = 0, i.e. the axes commute and the commutator vanishes. But, for space, we have that minus sign, i.j = -j.i, and [i.j] = 2k. That minus sign is critical for the representation of the *rotation*. Without it, we cannot rotate! This means we must modify the concept of "a vector." Hamilton recognized a vector as an entity with magnitude and direction; i.e. having just *two* properties. But, a vector, to him, was only formed from the three imaginary parts of the quaternion. These axes have a "property relation symmetry" among them, that permits of the exchange of any two axes and the structure will still have that ability to orient the hand and rotate within the space. But, if we were to swap out an imaginary axis, and replace it with the real scalar axis of the quaternion, the newly formed 3-space would lose this feature. In the special 4-dimensional space of the quaternion, therefore, we cannot simply exchange any two axes. Our 4-vector then, must have more than just the two properties of magnitude and direction. Each vector has a "property relation" with other vectors of the space, in particular, with the defining axis vectors, that restrict the kinds of operations that can be performed. For example, if we could rotate the scalar axis so that it becomes aligned with a space axis, then a previously commuting variable would become anti-commuting, when commutation is then measured relative to the remaining unaltered space axes. But, commutation is a discrete property, that does not admit of "degrees of commutation" between commute and anti-commute. So, such an operation is practically impossible. Time cannot become spacelike, contrary to some interpretations of Einstein's relativity, since that would imply that the non-orientable measures could become orientable measures, by a mere continuous transformation, and then acquire all the additional special properties of an orientable measure, such as admitting handedness and permitting rotations.

We must modify the concept of "vector," therefore, to recognize these "three" properties, of magnitude, direction, and orientability.

FOUR ELEMENTS: The ancient philosophers recognized the world as being constructed from four elements: Fire, Air, Water, and Earth. The condensation of a fifth element, some called the Aether, produced all four elements, so that it becomes possible to exchange the quantity of that fifth element among the four, and so change the proportions of the four elements that are present. That fifth element is not directly observable, however, it must be deduced from observations of the changes in the four observables. But, the four elements themselves combine to form another four elements on a denser level of existence, so that there are really Fires, Airs, Waters, and Earths, of differing densities on different planes or levels of manifest existence. The physical plane, which we are on, consists of the most dense manifestations of the four elements. The astral plane, where the spirit lives, is one step less dense than ours. The *Gospel of Thomas* says that "*If you do not fast from the world, you will not find*":27 this higher world of the spirit, since the senses are otherwise overloaded with the denser experiences that drown out the more subtle sensations of the higher realm. Once one has tuned out the noise of this lower world, however, and "*when you make eyes in place of an eye, a hand in place of a hand, a foot in place of a foot, an image in place of an image, then you will enter*":22; this being an exact duplicate of the physical body, just made with less dense material.

On every plane of existence, there are these four elements—*as above, so below*—observable to sentient beings within that plane, and every phenomenon that becomes manifest to the beings on that plane arises through changes in the ratios of these four elements. To an observer, therefore, his directly experienced universe is describable by four continuously changing observable parameters: the plasmic fire element, the gaseous element, the liquid element, and the solid element. All sentient experience occurs through modifications of these four parameters. These four elements have their corresponding *psychophysical* states, which are experienced within the mind of the observer, and by which these four elements get their properties and are recognized.

The fire element heats things up and makes things lighter; e.g. hot air rises. The air element flows and pushes things around; e.g. the wind blowing and bending the trees. The water element makes things feel heavy and fall downward; e.g. like rain, water falls. The solid element makes contact possible, through touching and pressing. Thus the four elements are experienced as rising, pushing, falling, and pressing. The Theravadin Buddhists use this description to explain how a man walks. To walk, lift the left foot, push it forward, drop it down, and press on the ground. Repeat with the right foot. Thus, walking is lifting, pushing, dropping, and pressing, or, in terms of the original elements, it's fire, air, water, earth. This particular progression of transformations of the four elements defines the action of "walking" on the physical plane. This enables a sentient being to move around within the same plane. To move between the planes of existence a different progression is required: e.g. earth, water, fire, air, moves the being from the physical plane into the astral plane. This progression is symbolized by the ritual Buddhist artifact called the stupa or chorten. It represents "death" on the denser physical plane, and "birth" into the less dense higher plane. That is, the process of dying is experienced as pressing, then feeling heavy, suddenly feeling lighter—the snap—and finally moving around on the new plane of existence. The same progression is experienced by meditators and out-of-body travelers, so that the "death" need not be permanent detachment from the physical plane. By reversing the progression one re-enters the physical plane or takes "birth" in the denser world. The magic of traversing within or between the planes is accomplished by merely "recollecting" the psychophysical experiences in the particular order of the progression required for the transformations desired. The importance of the "recollecting" is described in the *Tibetan Book of the Dead*. The skill of walking within or between the planes must be acquired by practice. The Theravadin begins by going for a walk in the woods, slowing down the actions involved in the walking action until he can concentrate his attention on the intermediate stages that mark the transitions of the elements involved. This "walking meditation," as it is otherwise called, is practiced until he has mastered the art of mentally following this action in sufficient detail that he can then consciously re-order the elemental transitions by changing the order of psychophysical sensations occurring in his mind; i.e. by merely "recollecting" the stages of walking in a different order. The physical human body is only designed to give automatic aid to walking about within the physical plane, and to voluntarily traverse between the planes requires special instruction on that specific art of mental concentration, although it may be acquired spontaneously within a specific life owing to the sentient being's practice in a previous existence; i.e. the being may "recollect" his previous knowledge. *The Yoga Sutras of Patanjali* IV:1 describes this acquisition of siddhi powers by birth as one of the five methods the skill is obtained.

Another progression makes up the Zodiac, where Aries, Taurus, Gemini, and Cancer, etc., correspond to fire, earth, air, and water, and define the experience of the seasons of time, development, growing old, aging, rot and decay; and the acquisition and development of knowledge and wisdom within the sentient being. The ages begin with fire, condensing to earth, i.e. spiritual beings taking on physical forms, coming into contact with new things then bring lots of movement and activity, finally ending in the flood. After the diluvian age, things begin again. Many cultures record the memory of the global flood in their myths.

QUATERNIONS: If the universe is describable by four measurable observable parameters, that exchange a fifth quantity among them in facilitating adjustment of their individual measures, then all phenomena can be modeled by 4×4 matrices which describe transformations of and among these four parameters, to the extent that the phenomena can be explained by "linear transformations." This is a first approximation, only, but a useful one nevertheless. Well, it just so happens, as will be shown in the papers included in this text, that every 4×4 matrix, T, can be represented by combinations of right handed, A, and left handed, B', quaternions: $T = AB' + AB' + AB' + \ldots + AB'$; and all linear transformation operations can thus be described by sequences of operations on quaternions. This means that, in a world constructed from four observable parameters, the quaternions have an unique role to play. They can completely describe the transformations in that world, when they are linear, and can possibly explain why the four parameters organize their transitions among themselves in order that there are such things as handedness and rotation, even perhaps time and space, and other peculiarities seen in our world, manifesting among the transitions. With this in mind, guiding underlying motivations, the study of quaternions is being undertaken. These papers represent the latest important results achieved thus far in the author's researches.

HEXPENTAQUATERNIONS : A Two-Hand Quaternion Algebra

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We consider the matrix method for solving simple linear quaternion equations, and demonstrate that such solutions often implicitly involve both left-hand and right-hand basis elements in the intermediate reckoning. This helps us to construct a more complete quaternion algebra, that simultaneously includes both hands, and expands on William Rowan Hamilton's original four dimensional idea. We call this algebra Hexpentaquaternions—hexpe numbers—because it is constructed out of sixteen elements, naturally arranged into five four-dimensional subalgebras–two anti-commuting and three commuting—including all hypercomplex algebras built on the five groups of order eight. We then show how, by replacing right hand quaternions with left hand quaternions on the other side of variables, we can simplify and solve linear quaternion equations, using only elementary methods.

1. AN AMBIGUOUS HAND.

The square-roots of negative unity,

$$\sqrt{-1} = a.i + b.j + c.k \tag{1.1}$$

with i, j, k, the anti-commuting hypercomplex roots of -1, and a, b, c, elements of the real number set, with the property $(a^2 + b^2 + c^2) = 1$, are Hamilton's invention. In his system, the units { i, j, k} represent space dimensions, as a geometrical interpretation, within a four-dimensional number $q = q_0.1 + q_1.i + q_2.j + q_3.k$, where the scalar unit { 1} is simply an extra-dimensional parameter introduced to enable the algebra to possess most of the rather simple common rules of arithmetic as found in ordinary algebra.

These space units { i, j, k} obey the special product rules given by W. R. Hamilton in 1843[1] [1];

$$\begin{gathered} i^2 = j^2 = k^2 = -1 \\ i = jk = -kj, j = ki = -ik, k = ij = -ji \end{gathered} \tag{1.2}$$

But, in so defining the units, an ambiguity needs to be resolved, in regard to that geometric interpretation. Should the equation, $ij = +k$, indicate a right-hand rule, or a left-hand rule? Hamilton called this product right-hand, but in his application he defined the action to represent that the turning of a screw with a man's right hand, turning clockwise relative to the body as seen by him, would involve motion of the screw towards the body of the man [2] [2] . Today, scientists apply the exact opposite definition to Hamilton, for that same rotation motion called *right-hand*.

It is obvious that one could just as easily attach the algebraic form, $ij = +k$, to the geometric left-hand, as we now do to the geometric right-hand. A convention is required. Once the decision is made, however, then the alternate form, $ij = -k$, in algebra, would represent the corresponding opposite form, in geometry. Thus, by convention, $ij = +k$, is linked with the geometric right-hand. While, $ij = -k$, is linked with the geometric left-hand. In a right-hand system, the triple product, $ijk = -1$. While, in a left-hand system, this is, $ijk = +1$. It is sufficient, therefore, to give the sign on the unit of this triple product ijk to indicate the *handedness* established for the underlying basis. Those elements can have $ij = +k$ or $ij = -k$.

Writing down the equation (1.1), therefore, does not completely specify the solution to the problem of the square root of -1. The left-hand verses right-hand ambiguity for ijk still has to be resolved. In the context of real numbers, the square root of $+1$ has two possible solutions; $+1$ or -1. There is an ambiguity, which is usually resolved according to the specific nature of the problem whose solution is being modeled by the real algebra. In the context of complex numbers, the square root of -1 has again two solutions; $+i$ or $-i$. And again, the specifics of the problem being modeled by the complex algebraic equation resolves the ambiguity. But, in the context of quaternions, the square root of -1 has, not only the obvious two solutions, $+(a.i + b.j + c.k)$ or $-(a.i + b.j + c.k)$, that correspond to those real and complex cases, for a given triplet of real numbers (a, b, c), but quaternions are even more special, in that they also possess the additional unique dual-hand ambiguous nature inherent in relations among the basis elements.

Now, Hamilton specifies the hand of the basis elements, as part of his definition. And in this way, equation (1.1) appears to be unambiguous. The $ij = +k$. The elements form a right-hand system. The convention is established. But, this is the same kind of decision as mathematicians make when they define the square-root

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of a real number to always indicate the positive root—a useful tactic for some classes of problems, but too restrictive, and too limiting, for others. And this definition does not eliminate the actual ambiguity inherent in the solution to the general problem.

Indeed, given any particular triplet of real numbers (a, b, c), and told that $(a.i + b.j + c.k)$ is a square root of -1, we can immediately infer eight related roots with alternate $+-$ signs, not just two, that are solutions;

$$\begin{aligned} \sqrt{-1} &= +a.i + b.j + c.k \\ \sqrt{-1} &= +a.i + b.j - c.k \\ \sqrt{-1} &= +a.i - b.j + c.k \\ \sqrt{-1} &= +a.i - b.j - c.k \\ \sqrt{-1} &= -a.i + b.j + c.k \\ \sqrt{-1} &= -a.i + b.j - c.k \\ \sqrt{-1} &= -a.i - b.j + c.k \\ \sqrt{-1} &= -a.i - b.j - c.k \end{aligned} \tag{1.3}$$

We wouldn't just pick one, and discard all the others, because we recognize that the alternatives can sometimes form just as valid solutions as the initial triple (a, b, c) given. These solutions are the vertices of a cube inscribed within a sphere with unit radius. That cube itself, however, can be rotated about the origin giving rise to an infinite number of yet other solutions to the same problem—find the square root of -1. The multiplicity of roots is compactly recognized by simply saying, let (a, b, c) be variables, ranging over the unit sphere, so that $(a^2 + b^2 + c^2) = 1$, and all the possible roots indicated here then *seem* to be represented. This creates the feeling that equation (1.1) does indeed represent all possible solutions, and is thus complete. But still, we haven't really given recognition to all the possible roots, because this sphere is linked to a particular basis, a right-hand system, $ij = +k$, and there is yet another infinite set of solutions out there, linked to a left-hand system, $ij = -k$, that goes unrecognized by our biased one-hand convention.

By establishing the convention at the outset—that only the right-hand need be represented in the basis—when defining the quaternion system, we throw out a complete set of perfectly valid solutions to the same problem. Thus, we are then unable to say, that we have found the most complete set of solutions possible.

To remedy this situation, we shall deviate from Hamilton's decision, and instead give equal weight to both left-hand and right-hand basis elements.

We introduce subscripts `R` and `L` on the unit elements, to distinguish between those on the right-hand and those on the left-hand. So, $\{\ i_R, j_R, k_R\ \}$, will represent a right-hand basis, and $\{\ i_L, j_L, k_L\ \}$, will be our left-hand basis. Without the subscripts, $\{\ i, j, k\ \}$, the context will determine whether we're talking about the right, the left, or an ambiguous state where the hand is yet unspecified.

For the right-hand basis system $(ij = +k)$:

$$\begin{aligned} i_R^2 = j_R^2 = k_R^2 = -1, \qquad & i_R = +j_R k_R = -k_R j_R, \\ j_R = +k_R i_R = -i_R k_R, \qquad & k_R = +i_R j_R = -j_R i_R \end{aligned} \tag{1.4}$$

While for the left-hand basis system $(ij = -k)$:

$$\begin{aligned} i_L^2 = j_L^2 = k_L^2 = -1, \qquad & i_L = -j_L k_L = +k_L j_L, \\ j_L = -k_L i_L = +i_L k_L, \qquad & k_L = -i_L j_L = +j_L i_L \end{aligned} \tag{1.5}$$

Now, both systems are equally important to us. This naturally leads to the question of whether we could not simply construct an algebra which includes all seven elements $\{\ 1, i_R, j_R, k_R, i_L, j_L, k_L\}$ simultaneously, and thus obtain a bilateral quaternion algebra, inherently ambidextrous with regard to its geometric interpretation, being without the usual pre-established one-hand bias, and thus probably obtain a more useful and naturally correct system to work with.

Of course, this two-hand algebra, while it obviously includes at least two sub-algebras that are separately well defined—a left-hand quaternion system, and a right-hand quaternion system, sharing the same real scalar dimension—yet has those undefined cross-terms, between left and right basis elements, like $i_R j_L = ?$, still to be resolved, and so is not completely specified by the equations in (1.4) and (1.5) above. We need a way to establish the definitions of these cross-terms. One method of doing so suggests itself when attempting to solve linear quaternion equations with matrix algebra. And we shall take our hints from there to construct a more complete two-hand quaternion system, which, for reasons that will become obvious later, we shall call *Hexpentaquaternion Algebra.*

2. SOLVING SIMPLE LINEAR QUATERNION EQUATIONS.

Our objective now is to solve equations of the type,

$$A_1 q B_1 + A_2 q B_2 + ... + A_n q B_n = C \tag{2.1}$$

where $A_k, B_k, C,$ are known quaternion parameters (with k = 1, 2, ..., n), while q is the unknown quaternion variable whose value we have to determine.

Were we dealing with either real or complex algebra, this equation would be trivial to transform into the more elementary form,

$$Aq = C \tag{2.2}$$

with the new parameter A being given by,

$$A = A_1 B_1 + A_2 B_2 + ... + A_n B_n. \tag{2.3}$$

This is because the known parameters all commute with the unknown variable. Similarly, we could arrange the equation into the alternate simple form,

$$qB = C \tag{2.4}$$

where the new parameter B is again given by the same formula (2.3) that defines A, because for real and complex algebras it is always the case that $A = B$.

Once we have the equation in the simplified form, either (2.2) or (2.4), solution is elementary, being either,

$$A^{-1}Aq = A^{-1}C \tag{2.5}$$
$$1q = A^{-1}C \tag{2.6}$$
$$q = A^{-1}C \tag{2.7}$$

or,

$$qBB^{-1} = CB^{-1} \tag{2.8}$$
$$q1 = CB^{-1} \tag{2.9}$$
$$q = CB^{-1} \tag{2.10}$$

according to our re-arrangement. But, either way, the answer is the same, since we know, $B^{-1} = A^{-1}$, and the commuting property gives, $A^{-1}C = CA^{-1} = CB^{-1}$.

But, when dealing with quaternions, this simple method of arranging to simplify, and solving by multiplying by suitable inverse parameters, is complicated by the fact that the parameters don't commute.

If we could find a new parameter B'_k, so that $B'_k q = qB_k$, then we could use a similar re-arranging and simplifying procedure to that above to solve the quaternion equation, for then $A_1 q B_1 + ...$ could be written $A_1 B'_1 q + ...$, and all the known parameters again aggregate on one side of the unknown variable q, leading once more to the form $Aq = C$, which we can solve, or at least determine whether there are any solutions.

Consider, for example, the following simplified version of equation (2.1) for quaternions, with now just two factors, A and B,

$$Aq + qB = C \tag{2.11}$$

If we can indeed find that left-hand parameter B', one would be able to write,

$$Aq + B'q = C \tag{2.12}$$
$$(A + B')q = C \tag{2.13}$$
$$q = (A + B')^{-1}C \tag{2.14}$$

and the problem is solved for quaternions.

Alternatively, if we could find a new parameter A', so that $qA' = Aq$, we could also solve this quaternion equation, this time from the other side of the unknown variable,

$$qA' + qB = C \tag{2.15}$$
$$q(A' + B) = C \tag{2.16}$$
$$q = C(A' + B)^{-1} \tag{2.17}$$

again producing a solution.

What we need then, is to find an effective way to construct left-parameters that are equivalent in action to their corresponding right-parameters, and visa versa. This is not generally possible within a pure quaternion algebra. However, by changing the representation of the problem from quaternions to matrices, it then becomes possible to harness the additional algebraic features provided by matrix algebra to solve this problem. One must step out of pure quaternion algebra for some intermediate steps, that can only be done in matrix algebra, then return to quaternion algebra at the end to present the final solution within the quaternion structure again. This is somewhat reminiscent of a similar issue classical mathematicians faced with cubic equations and imaginary numbers. Some perfectly acceptable real valued roots could only be obtained by a process of reckoning that deviated into the domain of complex arithmetic in the intermediate steps, before ending up back in real arithmetic with acceptable solutions. There was no process to get these real valued cubic roots using only real arithmetic. Yet, they were perfectly good real valued solutions to certain cubic equations. We shall see some similarities to this peculiar situation here again.

We can define the elemental quaternions q, A, B, C;

$$q = q_0 + q_1 i + q_2 j + q_3 k \tag{2.18}$$
$$A = a_0 + a_1 i + a_2 j + a_3 k \tag{2.19}$$
$$B = b_0 + b_1 i + b_2 j + b_3 k \tag{2.20}$$
$$C = c_0 + c_1 i + c_2 j + c_3 k \tag{2.21}$$

either in terms of 4×1 column vectors of the form,

$$\begin{pmatrix} 1 \\ i \\ j \\ k \end{pmatrix}$$

so that,

$$q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}, A = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix}, B = \begin{pmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{pmatrix}, C = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix},$$

or in terms of their corresponding 1×4 row vectors. We shall choose the column vector approach to proceed with our matrix reckoning; and let us consider the ijk to form a RIGHT-HAND system here.

First, consider the product Aq,

$$Aq = (a_0 + a_1 i + a_2 j + a_3 k)\,(q_0 + q_1 i + q_2 j + q_3 k) \tag{2.22}$$

$$\begin{array}{ccccccccc} = & a_0 q_0 & + & a_0 q_1 i & + & a_0 q_2 j & + & a_0 q_3 k \\ + & a_1 i q_0 & + & a_1 i q_1 i & + & a_1 i q_2 j & + & a_1 i q_3 k \\ + & a_2 j q_0 & + & a_2 j q_1 i & + & a_2 j q_2 j & + & a_2 j q_3 k \\ + & a_3 k q_0 & + & a_3 k q_1 i & + & a_3 k q_2 j & + & a_3 k q_3 k \end{array}$$

resolve the binary products $ij = k$ etc..,

$$\begin{array}{ccccccccc} = & a_0 q_0 & + & a_0 q_1 i & + & a_0 q_2 j & + & a_0 q_3 k \\ + & a_1 q_0 i & - & a_1 q_1 & + & a_1 q_2 k & - & a_1 q_3 j \\ + & a_2 q_0 j & - & a_2 q_1 k & - & a_2 q_2 & + & a_2 q_3 i \\ + & a_3 q_0 k & + & a_3 q_1 j & - & a_3 q_2 i & - & a_3 q_3 \end{array}$$

re-arrange and write in 4×1 column vector form,

$$\begin{pmatrix} a_0 q_0 & - & a_1 q_1 & - & a_2 q_2 & - & a_3 q_3 \\ a_1 q_0 & + & a_0 q_1 & - & a_3 q_2 & + & a_2 q_3 \\ a_2 q_0 & + & a_3 q_1 & + & a_0 q_2 & - & a_1 q_3 \\ a_3 q_0 & - & a_2 q_1 & + & a_1 q_2 & + & a_0 q_3 \end{pmatrix}$$

factor into the equivalent matrix product form,

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & +a_0 & -a_3 & +a_2 \\ a_2 & +a_3 & +a_0 & -a_1 \\ a_3 & -a_2 & +a_1 & +a_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

this can then be re-written,

$$Aq = (a_0 \mathbf{1} + a_1 \boldsymbol{I} + a_2 \boldsymbol{J} + a_3 \boldsymbol{K})q \tag{2.23}$$

where,

$$\mathbf{1} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \boldsymbol{I} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}$$

$$\boldsymbol{J} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

Note that these matrices have $\boldsymbol{IJ} = +\boldsymbol{K}$, and obey all the product rules of a RIGHT-HAND quaternion system,

$$\boldsymbol{I}^2 = \boldsymbol{J}^2 = \boldsymbol{K}^2 = -\mathbf{1}, \quad \boldsymbol{I} = \boldsymbol{JK} = -\boldsymbol{KJ}, \\ \boldsymbol{J} = \boldsymbol{KI} = -\boldsymbol{IK}, \quad \boldsymbol{K} = \boldsymbol{IJ} = -\boldsymbol{JI}$$

so what we have is a complete representation for the RIGHT-HAND basis elements of Hamilton's quaternions,

$$1 \sim \mathbf{1}, \quad i_R \sim \boldsymbol{I}, \quad j_R \sim \boldsymbol{J}, \quad k_R \sim \boldsymbol{K}$$

using elementary 4×4 square matrices.

Now, consider the product qB,

$$qB = (q_0 + q_1 i + q_2 j + q_3 k)\,(b_0 + b_1 i + b_2 j + b_3 k) \tag{2.24}$$

$$\begin{array}{ccccccccc} = & q_0 b_0 & + & q_0 b_1 i & + & q_0 b_2 j & + & q_0 b_3 k \\ + & q_1 i b_0 & + & q_1 i b_1 i & + & q_1 i b_2 j & + & q_1 i b_3 k \\ + & q_2 j b_0 & + & q_2 j b_1 i & + & q_2 j b_2 j & + & q_2 j b_3 k \\ + & q_3 k b_0 & + & q_3 k b_1 i & + & q_3 k b_2 j & + & q_3 k b_3 k \end{array}$$

resolve the binary products $ij = k$ etc..,

$$\begin{array}{ccccccccc} = & q_0 b_0 & + & q_0 b_1 i & + & q_0 b_2 j & + & q_0 b_3 k \\ + & q_1 b_0 i & - & q_1 b_1 & + & q_1 b_2 k & - & q_1 b_3 j \\ + & q_2 b_0 j & - & q_2 b_1 k & - & q_2 b_2 & + & q_2 b_3 i \\ + & q_3 b_0 k & + & q_3 b_1 j & - & q_3 b_2 i & - & q_3 b_3 \end{array}$$

re-arrange and write in 4×1 column vector form,

$$\begin{pmatrix} q_0 b_0 & - & q_1 b_1 & - & q_2 b_2 & - & q_3 b_3 \\ q_0 b_1 & + & q_1 b_0 & + & q_2 b_3 & - & q_3 b_2 \\ q_0 b_2 & - & q_1 b_3 & + & q_2 b_0 & + & q_3 b_1 \\ q_0 b_3 & + & q_1 b_2 & - & q_2 b_1 & + & q_3 b_0 \end{pmatrix}$$

factor into the equivalent matrix product form,

$$\begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & +b_0 & +b_3 & -b_2 \\ b_2 & -b_3 & +b_0 & +b_1 \\ b_3 & +b_2 & -b_1 & +b_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

this can then be re-written,

$$qB = (b_0 \mathbf{1} + b_1 \boldsymbol{I}' + b_2 \boldsymbol{J}' + b_3 \boldsymbol{K}')q \tag{2.25}$$

where,

$$\mathbf{1} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \boldsymbol{I}' = \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\boldsymbol{J}' = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}, \boldsymbol{K}' = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

Note that these matrices have $\boldsymbol{I}'\boldsymbol{J}' = -\boldsymbol{K}'$, and obey all the product rules of a LEFT-HAND quaternion system,

$$\boldsymbol{I}'^2 = \boldsymbol{J}'^2 = \boldsymbol{K}'^2 = -\mathbf{1}, \quad \boldsymbol{I}' = -\boldsymbol{J}'\boldsymbol{K}' = \boldsymbol{K}'\boldsymbol{J}', \\ \boldsymbol{J}' = -\boldsymbol{K}'\boldsymbol{I}' = \boldsymbol{I}'\boldsymbol{K}', \quad \boldsymbol{K}' = -\boldsymbol{I}'\boldsymbol{J}' = \boldsymbol{J}'\boldsymbol{I}'$$

so what we have is a complete representation for the LEFT-HAND basis elements of Hamilton's quaternions,

$$1 \sim \mathbf{1}, \quad i_L \sim \boldsymbol{I}', \quad j_L \sim \boldsymbol{J}', \quad k_L \sim \boldsymbol{K}'$$

using elementary 4×4 square matrices.

Now let us revisit that equation introduced before,

$$Aq + qB = C \tag{2.11}$$

We see that by using matrices we can indeed collect the known parameters on one side of the q variable. Using the results above, we can now write this equation as,

$$\begin{pmatrix} a_0 & -a_1 & -a_2 & -a_3 \\ a_1 & +a_0 & -a_3 & +a_2 \\ a_2 & +a_3 & +a_0 & -a_1 \\ a_3 & -a_2 & +a_1 & +a_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} + \begin{pmatrix} b_0 & -b_1 & -b_2 & -b_3 \\ b_1 & +b_0 & +b_3 & -b_2 \\ b_2 & -b_3 & +b_0 & +b_1 \\ b_3 & +b_2 & -b_1 & +b_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

So, effectively, we have found, in a way, the left-parameter B', such that we may write,

$$Aq + B'q = C \tag{2.12}$$

and can solve the problem. But this is only accomplished by juggling representations of quaternions from elementary to matrix form and back. Moreover, we started out with one representation of quaternions in elementary basis format, and transformed the stated problem by introducing really two different matrix representations: (1) in column vector, and (2) in square matrix.

Considering the four parameters, A, B, C, q, we see that the two *factors*, A, B, are both changed into 4×4 square matrices, while the variable, q, and the inhomogeneous parameter, C, are re-written differently, as 4×1 column vectors. So, two different types of matrix representations are utilized simultaneously in the same problem. This contrasts with our original framing of the problem, where all four of these parameters are represented in the same $\{1, i, j, k\}$ basis element format.

Nevertheless, whether it is column vector or square, these are both matrix formats. Moreover, they are intricately related formats that are built around the particular dimension number 4, and so form just a subset of the more general $M \times N$ matrix algebra available to us.

Using matrix rules, we can now combine the factors to obtain,

$$\begin{pmatrix} a_0 + b_0 & -a_1 - b_1 & -a_2 - b_2 & -a_3 - b_3 \\ a_1 + b_1 & +a_0 + b_0 & -a_3 + b_3 & +a_2 - b_2 \\ a_2 + b_2 & +a_3 - b_3 & +a_0 + b_0 & -a_1 + b_1 \\ a_3 + b_3 & -a_2 + b_2 & +a_1 - b_1 & +a_0 + b_0 \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{pmatrix}. \tag{2.26}$$

The solution exists, provided that the combined matrix factor doesn't have a vanishing determinant. This is, however, the same simplified equation form, $Aq = C$, which we sought. Except now, that *new factor A* isn't a quaternion at all. The variable, q, and parameter, C, are still quaternions, just in 4×1 matrix format. But, the combined factor is a new kind of object to us. It's a matrix alright, but not *a quaternion in matrix format*. Were it a quaternion, we would be guaranteed of a solution, since every non-zero quaternion has a multiplicative inverse, so A^{-1} is guaranteed to exist, letting us complete the steps, $A^{-1}Aq = A^{-1}C$, and write $q = A^{-1}C$. Only when $A = 0$, are we unable to solve, and this only happens in the singular special case where all the components of A vanish, so $a_0 = a_1 = a_2 = a_3 = 0$.

But, we can't say the same thing here with our new type of A factor object. It's closer to a general square matrix with its eight components—half that of a full $4 \times 4 = 16$ matrix—instead of the usual 4 existing when writing quaternions as square matrices, and so can fail to have an inverse at times even when all its component values are non-zero. With this said, we are still able to find the solutions when they exist, and can always determine whether or not there is a solution at all. So, the problem is effectively solved with matrix algebra.

What is the additional apparatus that matrix algebra provides us with that enables us to solve these quaternion problems? Look again at the equations (2.23) and (2.25),

$$Aq = (a_0\mathbf{1} + a_1\boldsymbol{I} + a_2\boldsymbol{J} + a_3\boldsymbol{K})q = Aq \tag{2.27}$$

$$qB = (b_0\mathbf{1} + b_1\boldsymbol{I}' + b_2\boldsymbol{J}' + b_3\boldsymbol{K}')q = B'q \tag{2.28}$$

Equations (2.11), (2.12), (2.26), can actually be re-written,

$$((a_0 + b_0)\mathbf{1} + a_1\boldsymbol{I} + a_2\boldsymbol{J} + a_3\boldsymbol{K} + b_1\boldsymbol{I}' + b_2\boldsymbol{J}' + b_3\boldsymbol{K}')q = C \tag{2.29}$$

We see that the matrix method is effectively combining representations of both left-hand and right-hand basis elements simultaneously in the reckoning. *Matrix algebra then, allows us to incorporate the left-hand elements along with right-hand elements in the same equation.* The moving of the B parameter, in the term qB, to the other side of the variable q, so that we can write an equivalent result, $B'q$, in place, is accomplished by replacing the right-hand elements with corresponding left-hand elements, keeping the component values unchanged, the $B = b_0\mathbf{1} + b_1\boldsymbol{I} + b_2\boldsymbol{J} + b_3\boldsymbol{K}$, being replaced by $B' = b_0\mathbf{1} + b_1\boldsymbol{I}' + b_2\boldsymbol{J}' + b_3\boldsymbol{K}'$, and in this way the factor can be moved[3].

Within a right-hand basis alone, we are unable to move the known parameters to the other side of the unknown variable q. So, we can't simplify the equation to solve it the usual way. Matrices add that facility, allowing us to effectively construct left-parameter equivalents for corresponding right-parameters, and to aggregate all the knowns on one side of the variable to be found.

Thus, by changing the representation to matrices we are able to solve this problem, because the matrix algebra contains sufficient flexibility to allow for the expression of the left-hand elements together with the right-hand elements of the quaternion system. Hamilton's calculus is unable to cope, because it is limited to a right-hand system alone, lacking this required flexibility.

This suggests we might benefit from studying the algebra of those seven matrix elements, $\{\mathbf{1}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}, \boldsymbol{I}', \boldsymbol{J}', \boldsymbol{K}'\}$. These are effectively the same seven basis elements $\{1, i_R, j_R, k_R, i_L, j_L, k_L\}$, mentioned in our opening section. However, because of our particular representation, we can now use matrix algebra to find the results of the products that cross left-hand with right-hand, and so complete the picture for this extended quaternion system. We take the product of every element in our basis with every other element, and when a new matrix element is found, we add this to our set, and repeat the procedure, to get all possible matrices that can be obtained by taking products of existing elements. If a matrix, $\boldsymbol{Z}$, is in the set, we don't add the negative of this matrix when it shows up in a product, instead we'll represent this by an overall minus sign, $-\boldsymbol{Z}$, and consider the product already included.

There are then found to be a grand total of 16 matrix elements (TABLE T.1), which should come as no surprise, since the general 4×4 matrix has just this number of components, and hence degrees of freedom, and no more, yet is sufficiently able to solve all the problems posed by our equations. Of these sixteen elements, seven, of course, we've met before, and we'll refer to now by the labels[4], $\boldsymbol{E}, \boldsymbol{I}_R, \boldsymbol{J}_R, \boldsymbol{K}_R, \boldsymbol{I}_L, \boldsymbol{J}_L, \boldsymbol{K}_L$. The remaining nine, after some careful thought, we find somewhat appropriate to label, $\boldsymbol{I}_M, \boldsymbol{J}_M, \boldsymbol{K}_M, \quad \boldsymbol{I}_A, \boldsymbol{J}_A, \boldsymbol{K}_A, \quad \boldsymbol{I}_Z, \boldsymbol{J}_Z, \boldsymbol{K}_Z$; the latter nine forming 3 natural groups of triplets, one triplet being obtained by crossing right and left basis elements of the same axis-label—thus given the label M for MIDDLE-HAND elements—while the other two triplets are differentiated by the somewhat more arbitrarily chosen letters A and Z, just because we find it convenient to place these triplets *first* and *last* in the 16×16 product table (TABLE T.2), for visual symmetry. Let's explore these results.

$\boldsymbol{I}_M, \boldsymbol{J}_M, \boldsymbol{K}_M$. In the first new triplet, the matrices are formed from the product between right and left quaternion elements of similar axes, $I_M = I_R I_L = I_L I_R$ ... etc., so we have,

$$\boldsymbol{I}_M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad \boldsymbol{J}_M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{K}_M = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Note that $\boldsymbol{I}_M \boldsymbol{J}_M = \boldsymbol{J}_M \boldsymbol{I}_M = -\boldsymbol{K}_M$, and these matrices follow the product rules of the commutative system,

$$\boldsymbol{I}_M{}^2 = \boldsymbol{J}_M{}^2 = \boldsymbol{K}_M{}^2 = \mathbf{1}, \quad \boldsymbol{I}_M = -\boldsymbol{J}_M \boldsymbol{K}_M = -\boldsymbol{K}_M \boldsymbol{J}_M,$$
$$\boldsymbol{J}_M = -\boldsymbol{K}_M \boldsymbol{I}_M = -\boldsymbol{I}_M \boldsymbol{K}_M, \quad \boldsymbol{K}_M = -\boldsymbol{I}_M \boldsymbol{J}_M = -\boldsymbol{J}_M \boldsymbol{I}_M$$

Rather than being the roots of -1, these elements are the roots of $+1$, by contrast; and unlike the imaginary units in quaternions, these elements actually commute with each other. These are all diagonal matrices.

$\boldsymbol{I}_A, \boldsymbol{J}_A, \boldsymbol{K}_A$. In the second new triplet, the matrices are formed from the product between right and left quaternion elements of different axes, fixing the actual positions of R before L, while cyclically permuting the axis labels, so we get, $I_A = J_R K_L, J_A = K_R I_L, K_A = I_R J_L$, and we have,

$$\boldsymbol{I}_A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{J}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{K}_A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{I}_A \boldsymbol{J}_A = \boldsymbol{J}_A \boldsymbol{I}_A = -\boldsymbol{K}_A$, and these matrices follow the product rules of the commutative system,

$$\boldsymbol{I}_A{}^2 = \boldsymbol{J}_A{}^2 = \boldsymbol{K}_A{}^2 = \mathbf{1}, \quad \boldsymbol{I}_A = -\boldsymbol{J}_A \boldsymbol{K}_A = -\boldsymbol{K}_A \boldsymbol{J}_A,$$
$$\boldsymbol{J}_A = -\boldsymbol{K}_A \boldsymbol{I}_A = -\boldsymbol{I}_A \boldsymbol{K}_A, \quad \boldsymbol{K}_A = -\boldsymbol{I}_A \boldsymbol{J}_A = -\boldsymbol{J}_A \boldsymbol{I}_A$$

Rather than being the roots of -1, these elements are again the roots of $+1$; and again, unlike the imaginary units in quaternions, these elements actually commute with each other. The position of the non-zero component in either the first row or first column of the square matrix also determines the I, J, K, label assignment.

$\boldsymbol{I}_Z, \boldsymbol{J}_Z, \boldsymbol{K}_Z$. In the third new triplet, the matrices are once more formed from the product between right and left quaternion elements of different axes, fixing the actual positions of R before L, while this time permuting the axis labels acyclically, so that, $I_Z = K_R J_L, J_Z = I_R K_L, K_Z = J_R I_L$, so we have,

$$\boldsymbol{I}_Z = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad \boldsymbol{J}_Z = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\boldsymbol{K}_Z = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{I}_Z \boldsymbol{J}_Z = \boldsymbol{J}_Z \boldsymbol{I}_Z = -\boldsymbol{K}_Z$, and these matrices follow the product rules of the commutative system,

$$\boldsymbol{I}_Z{}^2 = \boldsymbol{J}_Z{}^2 = \boldsymbol{K}_Z{}^2 = \mathbf{1}, \quad \boldsymbol{I}_Z = -\boldsymbol{J}_Z \boldsymbol{K}_Z = -\boldsymbol{K}_Z \boldsymbol{J}_Z,$$
$$\boldsymbol{J}_Z = -\boldsymbol{K}_Z \boldsymbol{I}_Z = -\boldsymbol{I}_Z \boldsymbol{K}_Z, \quad \boldsymbol{K}_Z = -\boldsymbol{I}_Z \boldsymbol{J}_Z = -\boldsymbol{J}_Z \boldsymbol{I}_Z$$

So, once again, rather than being the roots of -1, these elements are roots of $+1$; and unlike the imaginary units in quaternions these elements actually commute with each other. The position of the non-zero component in either the first row or first column of the square matrix also determines the I, J, K, label assignment; but this time, all the four non-zero components in each of the square matrices are -1.

Some brief remarks on these imaginary units are probably in order here. First of all, we have obviously found sets of elements that represent two different types of roots: (1) the two roots of -1, given by the `R` and `L` anti-commuting elements; (2) the three roots of $+1$, given by the `M, A, Z`, commuting elements. These anti-commuting and commuting elements work together to complete the algebra. We can't just have an algebra with all the imaginary square roots being of -1 alone, we must include these three extra imaginary square roots of $+1$ to complete the picture.

We may notice that our definitions of the `A` and `Z` are somewhat similar to the definitions for `R` and `L`, respectively, in that both `R` and `A` are defined by *cyclically* permuting the axes labels IJK, while both `L` and `Z` are defined contrarily, by *acyclically* permuting the same[5].

$$I_R = J_R K_R, \quad I_A = J_R K_L, \quad \text{cyclical IJK} \tag{2.30}$$
$$I_L = K_L J_L, \quad I_Z = K_R J_L, \quad \text{acyclical IJK} \tag{2.31}$$

This might tempt us to call `A` and `Z` the RIGHT-HAND and LEFT-HAND roots of $+1$. This would give us a somewhat more pleasing symmetry— the square root of -1 being observed to have RIGHT-HAND and LEFT-HAND anti-commuting roots, with the square root of $+1$ taking on RIGHT-HAND, MIDDLE-HAND, and LEFT-HAND commuting roots by comparison. These anti-commuting doublet and commuting triplet may even remind us of fermions and bosons in particle physics, with the spin-half and spin-one characteristics of doublet $(-1/2, +1/2)$ and triplet $(-1, 0, +1)$ seeming to form a somewhat familiar parallel to this algebraic symmetry.

Whatever case might be made for the parallels with physics, however, the problem with calling `A` and `Z` the RIGHT-HAND and LEFT-HAND roots, is that these two triplets form isomorphic four-dimensional algebras with the MIDDLE-HAND. The three algebras based on $\{\boldsymbol{E}, \boldsymbol{I}_M, \boldsymbol{J}_M, \boldsymbol{K}_M\}$, $\{\boldsymbol{E}, \boldsymbol{I}_A, \boldsymbol{J}_A, \boldsymbol{K}_A\}$, and $\{\boldsymbol{E}, \boldsymbol{I}_Z, \boldsymbol{J}_Z, \boldsymbol{K}_Z\}$, are all interchangeable with each other, being algebraically indistinguishable. They form identical structures, or rather, they happen to be just different representations of the same one algebra. This particular algebra is given by the commuting rules,

$$\begin{aligned} &E^2 = +E, \\ &I^2 = J^2 = K^2 = E, \quad IJ = JI = -K, \\ &JK = KJ = -I, \quad KI = IK = -J. \end{aligned} \tag{2.32}$$

This algebra bears little resemblance to Hamilton's quaternion algebra. The algebra also differs from the previously studied Davenport [1] (1991[6], 1996[7]) commuting hypercomplex algebra, which is defined by,

$$\begin{aligned} &E^2 = +E, \\ &I^2 = J^2 = -K^2 = -E, \quad IJ = JI = +K, \\ &JK = KJ = -I, \quad KI = IK = -J. \end{aligned} \tag{2.33}$$

Davenport mixes one square root of $+1$ and two square roots of -1, i.e. $(K^2 = +1, I^2 = -1, J^2 = -1)$, in the same four-dimensional algebra. This commuting hypercomplex algebra, by contrast, consists of just the roots of $+1$ only, leaving the roots of -1 to be dealt with separately and entirely by the quaternions.

The main point we wish to make here, however, is that there really is just one type of commuting hypercomplex algebra contained in the three—`M, A, Z`—sub-algebras. This is in stark contrast to the two quaternion algebras, given by the equations in (1.4) and (1.5) above, which are clearly distinct between RIGHT-HAND and LEFT-HAND forms[8].

One could argue then, that this new extended quaternion algebra we've constructed really consists of just *three* distinct sub-algebras—a pair of RIGHT-HAND and LEFT-HAND algebras that are the anti-commuting roots of -1, and a single MIDDLE-HAND algebra which is a commuting root of $+1$. This latter algebra appearing, however, in three of its representations—`M, A, Z`—making the overall number of sub-algebras *appear* to be *five* in number.

But, there is more to this picture. While it is true that these three commuting algebras are really identical to each other, from an *intra*-algebra point of view, and so are better considered representations of the same one algebra, these representations, however, interact differently with the `R` and `L` sub-algebras, and so can't really be substituted for each other when considering the products between different sub-algebraic elements. The *inter*-algebra interactions bring out the distinctions among the three—`M, A, Z`. Thus, in the context of our extended algebraic structure, we really do have to consider *five* sub-algebras—`R, L, M, A, Z`—to complete the picture.

Naming the Algebra. Because there are 16 elements in our new algebra, we prepend HEX, taken from the alternative name—HEXADECANIONS—the *five* sub-algebras that span the extended structure then suggest PENTA[9], and the fact that this new system is derived by extending Hamilton's ideas suggests QUATERNIONS—so, we call this system HEXPENTAQUATERNION ALGEBRA, or for a more convenient alternative short name—`hexpe numbers` [10].

The most general `hexpe number` we can write down is,

$$\begin{aligned} h &= h_0\mathbf{E} \\ &+ h_{R1}\mathbf{I_R} + h_{R2}\mathbf{J_R} + h_{R3}\mathbf{K_R} \\ &+ h_{L1}\mathbf{I_L} + h_{L2}\mathbf{J_L} + h_{L3}\mathbf{K_L} \\ &+ h_{M1}\mathbf{I_M} + h_{M2}\mathbf{J_M} + h_{M3}\mathbf{K_M} \\ &+ h_{A1}\mathbf{I_A} + h_{A2}\mathbf{J_A} + h_{A3}\mathbf{K_A} \\ &+ h_{Z1}\mathbf{I_Z} + h_{Z2}\mathbf{J_Z} + h_{Z3}\mathbf{K_Z} \end{aligned} \tag{2.34}$$

Then, using the definitions of the IJK matrices, we can re-write this number in the alternative square matrix form.

$$h = [a_{ij}] = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{2.35}$$

with,

$$\begin{aligned} a_{00} &= +h_0 - h_{M1} - h_{M2} - h_{M3} \\ a_{10} &= +h_{R1} + h_{L1} + h_{A1} - h_{Z1} \\ a_{20} &= +h_{R2} + h_{L2} + h_{A2} - h_{Z2} \\ a_{30} &= +h_{R3} + h_{L3} + h_{A3} - h_{Z3} \\ \\ a_{01} &= -h_{R1} - h_{L1} + h_{A1} - h_{Z1} \\ a_{11} &= +h_0 - h_{M1} + h_{M2} + h_{M3} \\ a_{21} &= +h_{R3} - h_{L3} - h_{A3} - h_{Z3} \\ a_{31} &= -h_{R2} + h_{L2} - h_{A2} - h_{Z2} \\ \\ a_{02} &= -h_{R2} - h_{L2} + h_{A2} - h_{Z2} \\ a_{12} &= -h_{R3} + h_{L3} - h_{A3} - h_{Z3} \\ a_{22} &= +h_0 + h_{M1} - h_{M2} + h_{M3} \\ a_{32} &= +h_{R1} - h_{L1} - h_{A1} - h_{Z1} \\ \\ a_{03} &= -h_{R3} - h_{L3} + h_{A3} - h_{Z3} \\ a_{13} &= +h_{R2} - h_{L2} - h_{A2} - h_{Z2} \\ a_{23} &= -h_{R1} + h_{L1} - h_{A1} - h_{Z1} \\ a_{33} &= +h_0 + h_{M1} + h_{M2} - h_{M3} \end{aligned} \tag{2.36}$$

We can invert these linear equations to express the `hexpe`-coefficients in terms of the a_{ij} components. This is most easily done by recognizing that the `hexpe` matrix bases contain all the information needed to invert these equations. One simply takes the four a_{ij} components that correspond to the non-zeros in any particular `hexpe` basis matrix, multiplies them by the $+1$ or -1 that appears in the corresponding locations, adds these four terms up, and divides the total by 4. This yields the correct value for the coefficient corresponding to that particular `hexpe` basis matrix.

For example, say we wish to find the coefficient of $\boldsymbol{J}_A$, in equation (2.34), we look at its matrix components,

$$\boldsymbol{J}_A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

this tells us that we only need the four a_{ij} components—$\{a_{20}, a_{31}, a_{02}, a_{13}\}$—and we'd then multiply by the corresponding signed units—$\{+1, -1, +1, -1\}$—add and divide by 4 to give us,

$$h_{A2} = (a_{20} - a_{31} + a_{02} - a_{13})/4$$

In this way, we can easily write down the `hexpe` coefficients for any general 4×4 square matrix, by simply locating the non-zero components of the basis matrices.

Inverting the equations in (2.36) then gives us,

$$\begin{aligned} h_0 &= (+a_{00} + a_{11} + a_{22} + a_{33})/4 \\ \\ h_{M1} &= (-a_{00} - a_{11} + a_{22} + a_{33})/4 \\ h_{M2} &= (-a_{00} + a_{11} - a_{22} + a_{33})/4 \\ h_{M3} &= (-a_{00} + a_{11} + a_{22} - a_{33})/4 \\ \\ h_{A1} &= (+a_{10} + a_{01} - a_{32} - a_{23})/4 \\ h_{A2} &= (+a_{20} - a_{31} + a_{02} - a_{13})/4 \\ h_{A3} &= (+a_{30} - a_{21} - a_{12} + a_{03})/4 \\ \\ h_{Z1} &= (-a_{10} - a_{01} - a_{32} - a_{23})/4 \\ h_{Z2} &= (-a_{20} - a_{31} - a_{02} - a_{13})/4 \\ h_{Z3} &= (-a_{30} - a_{21} - a_{12} - a_{03})/4 \\ \\ h_{R1} &= (+a_{10} - a_{01} + a_{32} - a_{23})/4 \\ h_{R2} &= (+a_{20} - a_{31} - a_{02} + a_{13})/4 \\ h_{R3} &= (+a_{30} + a_{21} - a_{12} - a_{03})/4 \\ \\ h_{L1} &= (+a_{10} - a_{01} - a_{32} + a_{23})/4 \\ h_{L2} &= (+a_{20} + a_{31} - a_{02} - a_{13})/4 \\ h_{L3} &= (+a_{30} - a_{21} + a_{12} - a_{03})/4 \end{aligned} \tag{2.37}$$

Now that we can write the `hexpe number` as a general square matrix, and convert any 4×4 matrix back into `hexpe number` format, we can proceed to use our knowledge of matrix algebra to work out the corresponding calculus for the `hexpe` system.

In particular, we can now find the multiplicative inverse of any arbitrary `hexpe number` of the form given in equation (2.34), by using our knowledge of how to construct an inverse matrix of the form given in (2.35).

Linear Independence. The 16 `hexpe` basis matrices in (2.34) are easily shown to be linearly independent. If we could express any basis matrix in terms of the others then there would exist a set of values for the `hexpe` coefficients $\{h_0, h_{R1}, h_{R2}, ..., h_{Z3}\}$, not all 0, for which we'd have $h = 0$. But, $h = 0$ means that all a_{ij} components in (2.35) must be 0, and from the inverse equations (2.37), we see immediately, that all the `hexpe` coefficients must vanish also. So, there is no set of non-zero `hexpe` coefficients for which we can write $h = 0$, hence the basis matrices are linearly independent.

Multiplicative Inverse. In general, to construct the inverse of a matrix like $[a_{ij}]$ given in (2.35), we need to calculate the overall determinant, $\mathtt{det}([a_{ij}])$, strike out the `i-th` row and `j-th` column in our 4×4 matrix to get the reduced 3×3 matrix whose determinant is called the Minor, M_{ij}, create the cofactor matrix, $[F_{ij}]$, by taking these Minors as components with alternating signs, $F_{ij} = (-1)^{i+j} M_{ij}$, transpose the cofactor matrix to obtain the adjoint matrix, $[F_{ji}] = [F_{ij}]^T$, and divide by that overall determinant to obtain the inverse, $[a_{ij}]^{-1} = [F_{ji}]/\mathtt{det}([a_{ij}])$. But, before proceeding with this method, let us examine the multiplicative inverses for a few special case `hexpe` numbers, using some alternate algebraic tricks.

`THE R-HAND`. First let us consider `hexpe` numbers that are just equivalent to the already familiar right-hand quaternions. All `hexpe` coefficients vanish, therefore, except $\{h_0, h_{R1}, h_{R2}, h_{R3}\}$. Our number then looks like;

$$h = h_0\mathbf{E} + h_{R1}\mathbf{I_R} + h_{R2}\mathbf{J_R} + h_{R3}\mathbf{K_R} \tag{2.38}$$

We know that the product of the imaginary basis elements anti-commute, $\boldsymbol{I}_R\boldsymbol{J}_R = -\boldsymbol{J}_R\boldsymbol{I}_R$, so that if we multiply h by itself, there will be cross terms like $\boldsymbol{I}_R\boldsymbol{J}_R + \boldsymbol{J}_R\boldsymbol{I}_R$ which cancel, but also other cross terms like $\boldsymbol{E}\boldsymbol{I}_R + \boldsymbol{I}_R\boldsymbol{E}$, that will leave us with some imaginary components remaining in our result. To get the latter type of cross terms to vanish also, we change the sign on the imaginary components (or alternatively change the sign on the $\boldsymbol{E}$), and so construct a new number defined by,

$$g = h_0\mathbf{E} - h_{R1}\mathbf{I_R} - h_{R2}\mathbf{J_R} - h_{R3}\mathbf{K_R} \tag{2.39}$$

Now, when we take the product, gh, both types of cross terms will vanish, in our result, leaving us with a number just proportional to $\boldsymbol{E}$.

$$gh = (h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2)\boldsymbol{E} \tag{2.40}$$

Our inverse, h^{-1}, then, is obtained by dividing the new number, g, by the factor[11] that makes the R.H.S the unit matrix.

$$h^{-1} = \frac{h_0\mathbf{E} - h_{R1}\mathbf{I_R} - h_{R2}\mathbf{J_R} - h_{R3}\mathbf{K_R}}{h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2} \tag{2.41}$$

This simple transformation is so useful, that we typically give the changing of signs a special name, the '`conjugate`,' and use an asterix to write, $h^* \equiv g$.

`THE L-HAND`. Next let us consider `hexpe` numbers equal to the left-hand quaternions. Now, all `hexpe` coefficients vanish except $\{h_0, h_{L1}, h_{L2}, h_{L3}\}$. Our number is,

$$h = h_0\mathbf{E} + h_{L1}\mathbf{I_L} + h_{L2}\mathbf{J_L} + h_{L3}\mathbf{K_L} \tag{2.42}$$

The situation is very similar to the right-hand quaternions. The product anti-commutes, $\boldsymbol{I}_L\boldsymbol{J}_L = -\boldsymbol{J}_L\boldsymbol{I}_L$. So, we get vanishing cross terms when we multiply h by itself. To get even the $\boldsymbol{E}\boldsymbol{I}_L + \boldsymbol{I}_L\boldsymbol{E}$ terms to vanish, we play the same trick of changing signs to define,

$$g = h_0\mathbf{E} - h_{L1}\mathbf{I_L} - h_{L2}\mathbf{J_L} - h_{L3}\mathbf{K_L} \tag{2.43}$$

Then the product, gh, is again proportional to the unit matrix.

$$gh = (h_0^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2)\boldsymbol{E} \tag{2.44}$$

And our inverse, h^{-1}, is determined in the same manner, by dividing the new number, g, by the factor that makes the R.H.S equal to the unit matrix.

$$h^{-1} = \frac{h_0\mathbf{E} - h_{L1}\mathbf{I_L} - h_{L2}\mathbf{J_L} - h_{L3}\mathbf{K_L}}{h_0^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2} \tag{2.45}$$

We can define the `conjugate` here also, representing that same kind of sign change transformation we met with in the right hand case, and write, $h^* \equiv g$. So, this concept of the `conjugate` is applicable to both right-hand and left-hand imaginary elements, and helps us to construct the multiplicative inverse of a number in both cases.

`THE M-HAND`. Now we'd like to consider the middle-hand numbers. The inverses for these `hexpe` numbers are somewhat more complicated. In the case where all the `hexpe` coefficients vanish except $\{h_0, h_{M1}, h_{M2}, h_{M3}\}$, the number becomes,

$$h = h_0\mathbf{E} + h_{M1}\mathbf{I_M} + h_{M2}\mathbf{J_M} + h_{M3}\mathbf{K_M} \tag{2.46}$$

This time these imaginary elements commute, however, we have, $\boldsymbol{I}_M\boldsymbol{J}_M = \boldsymbol{J}_M\boldsymbol{I}_M$, so multiplying h by itself would leave quite a few imaginary terms floating around the result. To get rid of as many of these as possible we must change the sign on only two imaginary elements, leaving the third with its original sign, so that we'd get the maximum number of cross terms canceling each other. Therefore we first construct the number,

$$g = h_0\mathbf{E} + h_{M1}\mathbf{I_M} - h_{M2}\mathbf{J_M} - h_{M3}\mathbf{K_M} \tag{2.47}$$

then observe that the product, gh, is,

$$\begin{aligned} gh &= (h_0^2 + h_{M1}^2 - h_{M2}^2 - h_{M3}^2)\boldsymbol{E} \\ &+ (2h_0h_{M1} + 2h_{M2}h_{M3})\boldsymbol{I}_M \end{aligned} \tag{2.48}$$

This result has the form, $(a\boldsymbol{E}+b\boldsymbol{I}_M)$, and we can use the standard algebraic formula, $(x^2-y^2)=(x+y)(x-y)$, to find our inverse. We first define a new factor, f, to have the complementary form, $(a\boldsymbol{E}-b\boldsymbol{I}_M)$, so,

$$\begin{aligned} f &= (h_0^2+h_{M1}^2-h_{M2}^2-h_{M3}^2)\boldsymbol{E} \\ &- (2h_0h_{M1}+2h_{M2}h_{M3})\boldsymbol{I}_M \end{aligned} \quad (2.49)$$

Then because, $(a\boldsymbol{E}-b\boldsymbol{I}_M)(a\boldsymbol{E}+b\boldsymbol{I}_M)=(a^2\boldsymbol{E}^2-b^2\boldsymbol{I}_M{}^2)$, and, $\boldsymbol{I}_M{}^2=\boldsymbol{E}^2=\boldsymbol{E}$, the product, fgh, would give us,

$$fgh = (a^2-b^2)\boldsymbol{E} \quad (2.50)$$

where

$$a = (h_0^2+h_{M1}^2-h_{M2}^2-h_{M3}^2) \quad (2.51)$$

$$b = (2h_0h_{M1}+2h_{M2}h_{M3}) \quad (2.52)$$

Our inverse, h^{-1}, is then obtained by dividing the product, fg, by this normalizing factor, (a^2-b^2), so that,

$$h^{-1}=\frac{w_0\boldsymbol{E}+w_1\boldsymbol{I}_M+w_2\boldsymbol{J}_M+w_3\boldsymbol{K}_M}{a^2-b^2} \quad (2.53)$$

where

$$\begin{aligned} a^2-b^2 &= h_0^4+h_{M1}^4+h_{M2}^4+h_{M3}^4 \\ &- 2h_0^2h_{M1}^2-2h_0^2h_{M2}^2-2h_0^2h_{M3}^2 \\ &- 2h_{M2}^2h_{M3}^2-2h_{M1}^2h_{M3}^2-2h_{M1}^2h_{M2}^2 \\ &- 8h_0h_{M1}h_{M2}h_{M3} \end{aligned} \quad (2.54)$$

and,

$$\begin{aligned} w_0 &= h_0^3-h_0(h_{M1}^2+h_{M2}^2+h_{M3}^2)-2h_{M1}h_{M2}h_{M3} \\ w_1 &= h_{M1}^3-h_{M1}(h_0^2+h_{M2}^2+h_{M3}^2)-2h_0h_{M2}h_{M3} \\ w_2 &= h_{M2}^3-h_{M2}(h_{M1}^2+h_0^2+h_{M3}^2)-2h_{M1}h_0h_{M3} \\ w_3 &= h_{M3}^3-h_{M3}(h_{M1}^2+h_{M2}^2+h_0^2)-2h_{M1}h_{M2}h_0 \end{aligned}$$

The inverse, h^{-1}, obviously doesn't exist when $a^2-b^2=0$, but otherwise it is defined by the formulas in (2.53-54).

Consider, for example, the special case obtained when, $\{h_0=1, h_{M1}=-1, h_{M2}=1, h_{M3}=1\}$, our number is,

$$h=\boldsymbol{E}-\boldsymbol{I}_M+\boldsymbol{J}_M+\boldsymbol{K}_M \quad (2.55)$$

We see that, $a=0, b=0$, so that $a^2-b^2=0$, and our denominator vanishes in (2.53), so this number has no inverse. We might notice that the w_k all vanish also, for these particular coefficients, and might think that $w_k/(a^2-b^2)=0/0$, being undefined, suggests that it's possible for an inverse to exist anyway. After all, if the numerator vanishes because of a proportional zero factor, e.g. $w_k=u_k(a^2-b^2)$, then although both numerator and denominator vanish independently, according to our formulas there could be situations where the ratio was nevertheless finite, e.g. $w_k/(a^2-b^2)=u_k$. But we note that if we constructed the g number, by changing signs, we'd get,

$$g=\boldsymbol{E}-\boldsymbol{I}_M-\boldsymbol{J}_M-\boldsymbol{K}_M \quad (2.56)$$

and we observe that the product of these two numbers now vanishes, $gh=0$.

Now, if a finite non-zero inverse did exist, say h^{-1}, then because the numbers all commute here we would be able to write, $h^{-1}h=hh^{-1}=E$, and multiplying by g we'd obtain $ghh^{-1}=gE=g$. The associative law for matrices, $(xy)z=x(yz)$, then guarantees that we may compute this in either of two ways, $(gh)h^{-1}=g(hh^{-1})$. But the L.H.S is $0h^{-1}=0$, while the R.H.S is $gE=g$, so we'd get $g=0$, which contradicts the definition (2.56). So, there can be no multiplicative inverse for our h number. In other words, given any particular number, h, the existence of a non-zero factor, g, that produces zero on taking the product, $gh=0$, is sufficient evidence that the number, h, has no multiplicative inverse. From equation (2.50), we see that whenever, $(a^2-b^2)=0$, we are able to construct such a null producing factor.

To see this, note that when $fgh=0$, this means that exactly one of the following conditions must hold:

(1) $f=0$;
(2) $f\neq 0, fg=0$;
(3) $f\neq 0, fg\neq 0$;

In case (1), if $f=0$, then since $f=a\boldsymbol{E}-b\boldsymbol{I}_M$, we must have, $a=b=0$—because the basis matrices are linearly independent—so that $gh=a\boldsymbol{E}+b\boldsymbol{I}_M=0$ also, and since we know $h\neq 0$ and $g\neq 0$, we've found the non-zero factor, g, that has a null product with h. In case (3), it follows, immediately, that the product, fg, itself, is that non-zero factor which has a null product with h. In case (2), things are a bit more involved. From the definitions of g and f we can write,

$$g=(2h_0\boldsymbol{E}+2h_{M1}\boldsymbol{I}_M)-h \quad (2.57)$$

$$fg=(a\boldsymbol{E}-b\boldsymbol{I}_M)(2h_0\boldsymbol{E}+2h_{M1}\boldsymbol{I}_M)-fh \quad (2.58)$$

We know $f\neq 0$ is given, so either $a\neq 0$ or $b\neq 0$, or both these parameters are non-zero, but equation (2.50) still holds, so we must have $a^2=b^2$, and so both $a\neq 0$ and $b\neq 0$, simultaneously. Moreover, either $a=+b$ or $a=-b$, so either $f=a(\boldsymbol{E}-\boldsymbol{I}_M)$ or $f=a(\boldsymbol{E}+\boldsymbol{I}_M)$. Now since $a\neq 0$, but $fg=0$, we must have either $(\boldsymbol{E}-\boldsymbol{I}_M)g=0$ or $(\boldsymbol{E}+\boldsymbol{I}_M)g=0$, that is to say, either $g=\boldsymbol{I}_Mg$ or $g=-\boldsymbol{I}_Mg$. Comparing terms in the expressions,

$$g=h_0\boldsymbol{E}+h_{M1}\boldsymbol{I}_M-h_{M2}\boldsymbol{J}_M-h_{M3}\boldsymbol{K}_M \quad (2.59)$$

$$\boldsymbol{I}_Mg=h_{M1}\boldsymbol{E}+h_0\boldsymbol{I}_M+h_{M3}\boldsymbol{J}_M+h_{M2}\boldsymbol{K}_M \quad (2.60)$$

we see, either $\{h_0=h_{M1}, h_{M3}=-h_{M2}\}$, or alternatively, $\{h_0=-h_{M1}, h_{M3}=+h_{M2}\}$. So, we can re-write the g parameter,

$$g=h_0\boldsymbol{E}+h_0\boldsymbol{I}_M+h_{M3}\boldsymbol{J}_M-h_{M3}\boldsymbol{K}_M \quad (2.61)$$

$$\textit{or} \quad g=h_0\boldsymbol{E}-h_0\boldsymbol{I}_M-h_{M3}\boldsymbol{J}_M-h_{M3}\boldsymbol{K}_M \quad (2.62)$$

Then, since the g parameter is really constructed from the original h number by flipping a pair of signs, that original number must have the form,

$$h = h_0\boldsymbol{E} + h_0\boldsymbol{I}_M - h_{M3}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M \quad (2.63)$$

$$\textit{or} \quad h = h_0\boldsymbol{E} - h_0\boldsymbol{I}_M + h_{M3}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M \quad (2.64)$$

These are the forms of the `hexpe` number, h, where our case (2) conditions apply. Now combining the above results, either we have, $\{a = b, h_{M1} = h_0\}$, or the alternative, $\{a = -b, h_{M1} = -h_0\}$, in which case equation (2.58) becomes,

$$fg = 2ah_0(\boldsymbol{E} - \boldsymbol{I}_M)(\boldsymbol{E} + \boldsymbol{I}_M) - fh \quad (2.65)$$

$$\textit{or} \quad fg = 2ah_0(\boldsymbol{E} + \boldsymbol{I}_M)(\boldsymbol{E} - \boldsymbol{I}_M) - fh \quad (2.66)$$

In either case, that first term on the R.H.S vanishes, e.g. $(\boldsymbol{E} - \boldsymbol{I}_M)(\boldsymbol{E} + \boldsymbol{I}_M) = (\boldsymbol{E}^2 - {\boldsymbol{I}_M}^2) = (\boldsymbol{E} - \boldsymbol{E}) = 0$, and we're left with the result that, $fg = -fh$. But, we're told that $fg = 0$, so we conclude, $fh = 0$. Then, since $f \neq 0$, we see that it is f, this time, that is the non-zero factor that makes a null product with h.

Therefore whenever $(a^2 - b^2) = 0$, there exists a non-zero factor, either g, f, or fg, that makes a null product with h.

So, the vanishing of this denominator term $(a^2 - b^2)$ in (2.53-54) is sufficient to indicate the non-existence of the multiplicative inverse, and we needn't worry about special situations arising where the numerator and denominator both vanish in some proportional way that might leave finite ratio values in our formulas to somehow implicate the existence of a valid inverse.

`THE A-HAND`. When our `hexpe` numbers are equivalent to the middle-hand `A`-numbers, then all coefficients vanish except $\{h_0, h_{A1}, h_{A2}, h_{A3}\}$, and the number becomes,

$$h = h_0\mathbf{E} + h_{A1}\mathbf{I_A} + h_{A2}\mathbf{J_A} + h_{A3}\mathbf{K_A} \quad (2.67)$$

Since the `A HAND` sub-algebra is isomorphic to the `M HAND` sub-algebra, the multiplicative inverse for this h number is given by the same kind of formulas (2.53-54). By substituting `A LABEL`s for the `M LABEL`s in the above formulas, we obtain the results for the `A HAND`.

`THE Z-HAND`. When our `hexpe` numbers are equivalent to the middle-hand `Z`-numbers, then all coefficients vanish except $\{h_0, h_{Z1}, h_{Z2}, h_{Z3}\}$, and the number becomes,

$$h = h_0\mathbf{E} + h_{Z1}\mathbf{I_Z} + h_{Z2}\mathbf{J_Z} + h_{Z3}\mathbf{K_Z} \quad (2.68)$$

Since the `Z HAND` sub-algebra is isomorphic to the `M HAND` sub-algebra, the multiplicative inverse for this h number is also given by the same kind of formulas (2.53-54). By substituting `Z LABEL`s for the `M LABEL`s in the above formulas, we obtain the results for the `Z HAND`.

The Weight Factors w_k. Perhaps one of the most intriguing things about the middle-hand inverse formula is the set of weight factors, w_k, that appear in the numerator of the coefficients. These are essentially cubic volume measures. The 3-dimensional volumetric measures that make up these factors have easily understood geometric interpretations.

We have volumes of 3 types of rectangular boxes involved. There's the cube—a box with all dimensions equal. Then there's the square faced cuboid—a box with at least two equal dimensions. And finally, there's the general cuboid—a box where all three dimensions may differ.

Consider, for example, equation (2.54)'s w_0 formula. First we have the term, h_0^3, which is clearly the volume of a cube with side length, h_0. Then there's the last term containing, $h_{M1}h_{M2}h_{M3}$, which is the volume of a general cuboid with dimensions along the IJK axes. In the middle term, we recognize that sum-of-squares formula, $h_{M1}^2 + h_{M2}^2 + h_{M3}^2$, from the `cubic diagonal` of a box. In fact, it's the same box whose volume appears in the last term. Such a box has several diagonals. Each face has two diagonals that cross each other at the center of the face. The sum of squares of two sides gives the measure of the length for each of these diagonals. But, there's also the `cubic diagonal` which crosses the volumetric center of the box itself. This diagonal's length is measured by the sum of three squares. The actual length, of course, is *the square root* of the sum of squares formula, so the expression here really describes an area, the area of a square drawn on the cubic diagonal. Then, the whole term, $h_0(h_{M1}^2 + h_{M2}^2 + h_{M3}^2)$, describes the volume of that box with one square face constructed on the cubic diagonal, and the third dimension being the box's height given by, h_0—i.e. our square faced cuboid.

Once the geometrical picture is recognized, these *cuboid weight factors* are easy to remember. Each of the weight factors has the same kind of formula,

$$\texttt{cube} - \texttt{square}\square\texttt{cuboid} - 2 \times \texttt{cuboid}.$$

One only needs to appropriately permute the subscripts on the parameters to get the weight factors for each axis in turn. But, now, the inverse of a middle-hand `hexpe` number is almost as easy to recall to mind as that for a right or left quaternion. For the denominator, $(a^2 - b^2)$, one can interpret the 4-space volumetric terms in (2.54), in a similar geometric way, or alternatively, observe that

$$a^2 - b^2 = \quad h_0w_0 + h_{M1}w_1 + h_{M2}w_2 + h_{M3}w_3 \quad (2.69)$$

Notice that, when the `hexpe` number is specialized to one of the five sub-algebras—`R, L, M, A, Z`—there are therefore essentially only two types of constructions for the multiplicative inverse. The quaternions, left or right, have an inverse constructed by taking the conjugate and dividing by the square norm. While, the middle-hand commutative hypercomplex sub-algebras have a somewhat more complicated inverse construction given by the cuboid weight factors divided by that sum.

Nulls. In equation (2.53), when the parameter $(a^2 - b^2)$ vanishes, the middle-hand hexpe number, h, has no inverse. Under these conditions, if $h \neq 0$, there exists another non-zero number, p, that makes a null product with h, i.e, $ph = 0$. Our interest now, is to locate the subdomain of the 4-dimensional space where this occurs. We can express this normalizing parameter in a few alternative ways;

$$a^2 - b^2 \;=\; (a-b)(a+b) \tag{2.70}$$

$$\begin{aligned} =\; & \left[(h_0^2 + h_{M1}^2 - h_{M2}^2 - h_{M3}^2) - (2h_0 h_{M1} + 2h_{M2}h_{M3})\right] \\ \times\; & \left[(h_0^2 + h_{M1}^2 - h_{M2}^2 - h_{M3}^2) + (2h_0 h_{M1} + 2h_{M2}h_{M3})\right] \end{aligned}$$

$$\begin{aligned} =\; & [h_0 - h_{M1} - h_{M2} - h_{M3}] \\ \times\; & [h_0 - h_{M1} + h_{M2} + h_{M3}] \\ \times\; & [h_0 + h_{M1} - h_{M2} + h_{M3}] \\ \times\; & [h_0 + h_{M1} + h_{M2} - h_{M3}] \end{aligned}$$

$$= a_{00}a_{11}a_{22}a_{33} \tag{2.71}$$

Essentially then, our normalizing parameter is just the product of the main diagonal components in the equivalent matrix, $[a_{ij}]$, from (2.35). When one or more of these four components vanish our denominator vanishes. To understand the situation better, we can exploit our familiarity with the usual vector algebra to get a more revealing geometric grasp of the conditions.

In vector algebra, a plane is defined by a vector normal to the plane. Let the unit normal be, $\boldsymbol{n} = \alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k}$, then if the points in the plane are, $\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, the equation of the plane is given by, $\boldsymbol{n}\cdot\boldsymbol{r} = d$, for some real valued number, d, that characterizes the plane.

$$(\alpha\boldsymbol{i} + \beta\boldsymbol{j} + \gamma\boldsymbol{k})\cdot(x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}) = d \tag{2.72}$$

$$\alpha x + \beta y + \gamma z = d \tag{2.73}$$

We can see that the vanishing of any one of the factors in our denominator results in the equation of a plane, there being four such planes with unit normals given by,

$$\begin{aligned} \boldsymbol{n}_1 &= (+\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})/\sqrt(3) \\ \boldsymbol{n}_2 &= (+\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k})/\sqrt(3) \\ \boldsymbol{n}_3 &= (-\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k})/\sqrt(3) \\ \boldsymbol{n}_4 &= (-\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})/\sqrt(3) \end{aligned} \tag{2.74}$$

There are different ways to view these planes. One convenient method is to align the IJK units of the M-H number with the $\boldsymbol{ijk}$ axes of vector algebra, then interpret the planes as being in the familiar 3-space, all characterized by the same, $d = h_0/\sqrt{3}$, parameter value, but with each having a different direction for the unit normal vector, $\boldsymbol{n}$.

As it turns out, these four planes all intersect each other at the same angle.

$$\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 = \cos(\theta) = -1/3 \tag{2.75}$$

$$\theta = \arccos(-1/3) = \pi - \arccos(1/3) \tag{2.76}$$

$$\theta = 109.4712^\circ = 109^\circ 28' 16'' \tag{2.77}$$

This angle, $\theta = 109.47^\circ$, is the dihedral angle of the `regular octahedron`. A *dihedral angle* is the angle between two intersecting plane faces of a given polyhedron. The `cube` has dihedral angle of 90°. The `regular tetrahedron` has dihedral angle of 70.53°. These three polyhedra are all closely related solids.

The cube has 6 faces, 8 vertices, and 12 edges, while the octahedron has 8 faces, 6 vertices, and 12 edges. So, we can pick the midpoints on the faces of the cube, join them up, and construct the octahedron. Or, pick the midpoints on the faces on the regular octahedron, join them up, and we'd get a cube. So, the cube and octahedron are called dual solids—from one we can construct the other. A major consequence of this is that they both have the same symmetry group. The group of transformations that leave the cube unchanged also leave the regular octahedron unchanged.

So, our middle-hand numbers have vanishing inverses on planes that intersect in octahedral fashion. To get a better picture of this, recall that the octahedron is made up of two five sided pyramids. There are intricate relationships between such types of pyramids and both the cube and octahedron. A cube, for example, can be broken out into six pyramids. Let the center of the cube be the apex of a pyramid and each cube face be the square base on a pyramid, then we can literally `invert the cube`, i.e. turn it inside out, and obtain six identical pyramids. These pyramids don't have the same face angles as the two in our octahedron. But they belong to the same class of five sided polyhedra. They all look like the Great Pyramid on the Giza plateau in Egypt, but with varying construction angles.

Now imagine that we pick up the Great Pyramid in Egypt and construct another such pyramid under it, upside down, so that we can join their bases together. Then we'd get an octahedron. It's not a `regular` octahedron, just a general octahedron. But now, instead of joining their bases, let us remove the lower pyramid and place it above the original pyramid, with its apex still pointing down, and join the two pyramids at their main apexes instead. This creates two funnels, with square cross-sections, leading away from the joined apex point in two opposite directions. Now elongate the funnels by extending each pyramid height, thus moving its square base out towards infinity, while still keeping the same face angles between the planes.

We now have forward and backward funnels, whose inside and outside regions define subspaces where the M-H inverse exists, these regions being bounded by the four planes where inverses aren't allowed, and null producing factors take their place. These funnels are somewhat like the cones in special relativity, except our funnels have square cross-sections, rather than the circular cross-sections in relativity, and except our outside region is also subdivided into compartments because those four planes continue to extend off to infinity also in other directions, and don't just stop at the corner edges of the funnel where they intersect.

The joint apex point where all four planes meet in a single point is defined by the equations, $h_{M1} = h_{M2} = h_{M3} = h_0$. In other words, in our parallel illustrative vector algebra, we could write,

$$\boldsymbol{r} = h_0(\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}) \tag{2.78}$$

which locates the apex in 3-space for a given value of the fourth parameter, h_0. If we now draw in the four unit normal vectors, $\boldsymbol{n}_1, \boldsymbol{n}_2, \boldsymbol{n}_3, \boldsymbol{n}_4$, given in (2.74), leading out from this point, those normals will define the locations of the four vertices of a `regular tetrahedron`, whose center is this funnel apex. So the 109.47° dihedral angle of the octahedron is also a key angle in the regular tetrahedron, except it's an internal angle here, the tetrahedron's own dihedral angle being the corresponding complement angle, $70.53° = 180° - 109.47°$.

Invariants. The unit right hand quaternions generate rotations which are rigid body changes that preserve the distances between points of an object, enabling it to keep its shape even while it is transformed about in 3-space. In the construction of the inverse, we must divide by that parameter, called the `'square norm'`, which, as it turns out, is the very expression that measures those distances that remain invariant.

$$N_R^2(w, x, y, z) = (w^2 + x^2 + y^2 + z^2) \tag{2.79}$$

The unit middle hand numbers generate transformations that preserve different quantities. No longer are these metric distances of the space preserved, the distances between points may increase or decrease after transformation. In the construction of the inverse here, we find we must now divide by a slightly more complicated `'quartic norm'`, which, we may write,

$$\begin{aligned} N_M^4(w, x, y, x) &= (w - x - y - z) \\ &\times\ (w - x + y + z) \\ &\times\ (w + x - y + z) \\ &\times\ (w + x + y - z) \end{aligned} \tag{2.80}$$

Let us say that our M-H number is the usual,

$$h = h_0\boldsymbol{E} + h_{M1}\boldsymbol{I}_M + h_{M2}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M \tag{2.81}$$

When we apply this transformation operator, h, to our quaternion 4-space point, $q = (w, x, y, z)$, this transforms the coordinates into the new point, $q' = (w', x', y', z')$.

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a_{00}w \\ a_{11}x \\ a_{22}y \\ a_{33}z \end{pmatrix} \tag{2.82}$$

$$w'x'y'z' = a_{00}a_{11}a_{22}a_{33} \cdot wxyz \tag{2.83}$$

The effect of this M-H number then, is to scale each coordinate variable by a different factor. This is usually referred to as a `nonproportional scaling` transformation, and it contrasts with the `proportional scaling` that occurs when multiplying by a R-H quaternion.

Notice, however, that when the M-H normalizing factor is 1, i.e. $N_M^4 = a_{00}a_{11}a_{22}a_{33} = 1$, the 4-space volume of the object remains unchanged in (2.83). Although the relative ratios of the coordinate measures may still change, $w'/x' = (a_{00}/a_{11}) \cdot (w/x)$, etc.. the 4-volume is now a preserved quantity—*the object changes shape while keeping its 4-volume fixed.*

For R-H quaternions, when the normalizing factor is one, i.e. $N_R^2 = 1$, we have the unit R-H quaternion, and preserved distances. Similarly, for M-H middle-hand numbers, when the normalizing factor is one, i.e $N_M^4 = 1$, we say we have the unit M-H middle-hand number, and preserved 4-volumes.

$$h_R' = \frac{h_0\boldsymbol{E} + h_{R1}\boldsymbol{I}_R + h_{R2}\boldsymbol{J}_R + h_{R3}\boldsymbol{K}_R}{\sqrt[2]{N_R^2(h_0, h_{R1}, h_{R2}, h_{R3})}} \tag{2.84}$$

$$h_M' = \frac{h_0\boldsymbol{E} + h_{M1}\boldsymbol{I}_M + h_{M2}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M}{\sqrt[4]{N_M^4(h_0, h_{M1}, h_{M2}, h_{M3})}} \tag{2.85}$$

We define the unit number by dividing by the root of the appropriate norm, so that, $h_R' \cdot h_R'^{*} = \boldsymbol{E}$, and, $h_M' \cdot h_M'^{\dagger} = \boldsymbol{E}$, where, $h_R'^{*}$, is the quaternion conjugate, and, $h_M'^{\dagger}$, is the middle-hand equivalent with cuboid weight factors in place of sign changes[12]. The norms, N_R^2 and N_M^4, are quite different expressions, because the transformations generated by the numbers are very different. Here, in the middle-hand numbers, the object does not rotate, but changes shape instead.

Shape Shifting. One curious observation about the Great Pyramid is that its truncated height is that of a cube with same volume as the pyramid. In other words, the Pharaoh stopped construction when he'd reached the height of a cube, not when he'd reached the height of a completed pyramid. To the untrained eye, there's only a five sided pyramid there, standing unfinished. But, to the trained mathematician, there is a cube sitting out there in the desert, with a shape distorted *to look like* a pyramid—it even has the six faces of the cube.

Now, the measurements of the Great Pyramid are not really known with sufficient precision to determine whether the cube is to be made equal in volume to the completed pyramid or the truncated pyramid. But, lets explore what is known about the measures. The missing capstone is 1/16-th of the pyramid's height to apex. So, if, H, is the pyramid's complete height, then the truncated height is, $15H/16$. Then let, $2L$, be the length of one side of the square base. The volume of the complete pyramid is, $V = 1/3 \cdot (2L)^2 \cdot H$, and the volume of the truncated pyramid is, $V' = (1 - (1/16)^3) \cdot 1/3 \cdot (2L)^2 \cdot H$.

CUBE-1. If we consider a cube with same volume as the completed pyramid, then, since the truncated height is the cube's height, we have,

$$\left(\frac{15H}{16}\right)^3 = \frac{(2L)^2 H}{3} \tag{2.86}$$

$$\left(\frac{H}{L}\right)^2 = \left(\left(\frac{16}{15}\right)^3 \frac{4}{3}\right) \tag{2.87}$$

$$= 1.61817283950617 \tag{2.88}$$

CUBE-2. If we consider a cube with the same volume as the truncated pyramid, then, since the truncated height is the cube's height, we have,

$$\left(\frac{15H}{16}\right)^3 = \left(1 - \left(\frac{1}{16}\right)^3\right) \frac{(2L)^2 H}{3} \tag{2.89}$$

$$\left(\frac{H}{L}\right)^2 = \left(\left(1 - \left(\frac{1}{16}\right)^3\right) \left(\frac{16}{15}\right)^3 \frac{4}{3}\right) \tag{2.90}$$

$$= 1.61777777777778 \tag{2.91}$$

PHI. Now there are alternative theories of the pyramid's construction design. The Golden Mean theory says that the area of a face is the same as the square on height,

$$H^2 = \frac{1}{2} \cdot (2L)(H^2 + L^2)^{1/2} \tag{2.92}$$

$$\tag{2.93}$$

$$\left(\frac{H}{L}\right)^2 = \left(\frac{1 + \sqrt{5}}{2}\right) = \phi \tag{2.94}$$

$$= 1.6180339887499 \tag{2.95}$$

PI. And then there's the circle theory, which claims the perimeter of the square base is equal to the circumference of the circle whose radius is the pyramid's height,

$$2\pi H = 4(2L) \tag{2.96}$$

$$\left(\frac{H}{L}\right)^2 = \left(\frac{4}{\pi}\right)^2 \tag{2.97}$$

$$= 1.6211389382774 \tag{2.98}$$

THEORY	$(H/L)^2$	H/L	BASE	ANGLE
pi-π	1.621139	1.273240	51.853974	$51°51'14''$
cube-1	1.618173	1.272074	51.828487	$51°49'43''$
phi-ϕ	1.618034	1.272020	51.827292	$51°49'38''$
cube-2	1.617778	1.271919	51.825088	$51°49'30''$

These theories are fairly close. But only the cube theory gives a rational for the missing capstone. The other PI and PHI theories have other merits, so perhaps the Pharaoh played with the numbers to build something inbetween all these various possibilities to indicate a connection between them. The difference between the two extreme values for base angle is less than $1\frac{3}{4}'$ of arc. That's too small for us to decide which theory is preferred. It isn't even clear whether we could decide the issue if the Great Pyramid was in better condition than it is today. There may have been deliberate errors built into the original pyramid design to make it impossible to determine its measures more accurately, just so that the pyramid could, in fact, implicate more than one theory simultaneously. The archeologist Piazzi Smyth observed that the King's chamber of the Great Pyramid seemed to have been constructed with deliberate built-in variances in the linear dimensions of the room, for some reason or the other, since it was obvious that the engineers were capable of constructing to much greater tolerances than is evident in the measurements. The base angle of the pyramid, which is the angle the triangular face makes with the square base, is usually quoted by pyramid theorists to indicate how close a particular theory comes to the facts of the case. It is usually near $51°50'40'' \pm 1'5''$, which is the archeologist W. M. Petrie's [13] measurement[1] on the North Face of the Great Pyramid.

`Khufu's Transform`. Our interest here, however, is in the cube theory. How would we mathematically describe such a shape shift? Since Khufu is the name of the Pharaoh usually credited with building the Great Pyramid, we'll refer to the operator that can change a cube into a truncated pyramid as `Khufu's Transform`.

Let's start with the cube, and set the origin of coordinates at its center. Let xy be the horizontal plane, and z the vertical. And let x be the north-south line, with south positive, y be the east-west line with east positive, and z positive in the up direction.

Our main interest is in the CUBE-2 theory. The cube is transformed into a truncated pyramid, with its volume unchanged, `and` its height unchanged. This means only the planes parallel to the xy horizontal plane experience scale changes. The square at the top of the cube shrinks in size, while the square at the base expands. The shrinking top and expanding bottom are so coordinated that they produce the exactly compensating scale shifts required to keep the volume invariant.

Now on each horizontal plane the scale changes in the x-axis and y-axis are the same, but the size of the scale factor changes with the vertical z axis. So, if (x, y, z) is a point on the surface of the initial cube, which becomes (x', y', z') after `Khufu's Transform`, we have,

$$x' = \alpha(z)x \tag{2.99}$$
$$y' = \alpha(z)y \tag{2.100}$$
$$z' = z \tag{2.101}$$

The faces of the Great Pyramid are actually slightly concave [1] [2] [3]. But this is an effect that is so small that it is invisible to the eye when looking from the ground. Only an aerial view at certain times of the day reveals the effect. So, we shall ignore this feature, and consider the faces flat, in which case our scale factor is only functionally dependent on the one z-coordinate, and can be considered a linear function, $\alpha(z) = az + b$. The transform equations become,

$$x' = (az + b)x \tag{2.102}$$
$$y' = (az + b)y \tag{2.103}$$
$$z' = z \tag{2.104}$$

Since we have only two parameters, $\{a, b\}$, we only need to consider how two points on the cube transform, to get the values of these parameters. Let's consider the mid-point, $(x_1, 0, z_1)$, on the edge of the top square, and the mid-point, $(x_2, 0, z_2)$, on the edge of the bottom square, where,

$$(x_1, 0, z_1) = (+\frac{15H}{32}, 0, +\frac{15H}{32}) \tag{2.105}$$
$$(x_2, 0, z_2) = (+\frac{15H}{32}, 0, -\frac{15H}{32}) \tag{2.106}$$

remembering that the height of the initial cube is the same as the height of the final truncated pyramid, which is $15H/16$, and our coordinate mid-points are therefore all 1/2 of this measure. Then, since, $2L$, is the expanded length of the side on the bottom square of the pyramid, and the top square shrinks to a size 1/16-th of that length, these coordinates transform to,

$$(x'_1, 0, z'_1) = (+\frac{L}{16}, 0, +\frac{15H}{32}) \tag{2.107}$$
$$(x'_2, 0, z'_2) = (+L, 0, -\frac{15H}{32}) \tag{2.108}$$

Putting these "before and after" coordinate values into our transform equations, and solving for the parameters, we obtain, $\{a, b\} = \{-32L/(15H^2), 17L/(15H)\}$, then, substituting these values, the equations become,

KHUFU'S TRANSFORM.

$$x' = \frac{L}{H} \cdot \left(\frac{17}{15} - 2 \cdot \frac{16}{15} \cdot \frac{z}{H}\right) x \tag{2.109}$$

$$y' = \frac{L}{H} \cdot \left(\frac{17}{15} - 2 \cdot \frac{16}{15} \cdot \frac{z}{H}\right) y \tag{2.110}$$

$$z' = z \tag{2.111}$$

Note that, in shape shifting the cube into the truncated pyramid, all horizontal planes undergo scale changes, except one plane which remains invariant. Setting, $x' = x$, to evaluate the position of this plane, we get,

$$z = \frac{1}{2} \cdot \frac{15H}{16} \cdot \left(\frac{17}{15} - \frac{H}{L}\right) \tag{2.112}$$

Since we established our origin of coordinates at the center of the cube, which is $15H/32$ higher than the bottom square, this position represents a height, H_0, from the base of the pyramid, given by,

$$H_0 = \frac{1}{2} \cdot \frac{15H}{16} \cdot \left(\frac{17}{15} - \frac{H}{L}\right) + \frac{1}{2} \cdot \frac{15H}{16} \tag{2.113}$$

$$= H \cdot \left(1 - \frac{1}{2} \cdot \frac{15}{16} \cdot \frac{H}{L}\right) \tag{2.114}$$

Given the value of our pyramid slope, this invariant plane turns out to be located below the center of the cube, and hence below the mid-point of the pyramid's truncated height, at a vertical distance of slightly more than the height of the missing capstone, $z = (H/16) \cdot (8.5 - 7.5 \cdot \sqrt{\phi}) \approx (H/16) \cdot (-1.040)$, and its height from base platform is, $H_0 \approx 0.4H$. On the Great Pyramid, this occurs at the 73-rd square platform from the ground, and is marked by a transition in the step size between the 73-rd and 74-th levels[2], giving the impression of a new pyramid base being formed at that plane[14]. Another jump in step size at 1/2 the truncated height, $\approx \frac{1}{2} \cdot 450ft = 2700in$, is marked by a strikingly suggestive stone step transition, where the under-step is 1/2 the upper-step, indicating that this is the point in the pyramid's height where the height from ground below to this point is 1/2 the total, i.e., suggesting that `the Great Pyramid is deliberately truncated by design`!

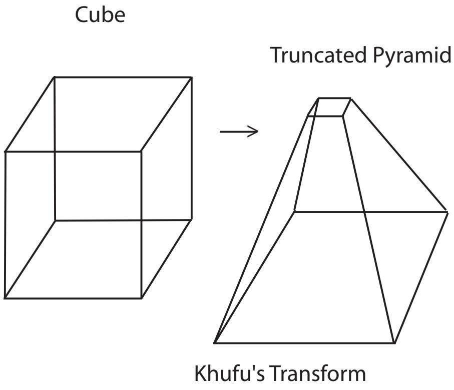

FIG. 1: The Great Pyramid

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Cuboid Scale Changes. Now, our M-H numbers generate shape shifts. These shape changes are, however, rather restricted. If we align a cube with its edges parallel to the coordinate axes, then the cube becomes a cuboid under this transformation. It never shape shifts to another type of polyhedron. Scale changes are strictly along axes lines, and so for the linear transformation there's one fixed scale change for each coordinate axis. The cube can never undergo the type of shape shift found in the Great Pyramid, which requires a varying scale change along a coordinate axis, and this means Khufu's Transform can't be implemented by simply multiplying by an M-H number, unless we allow the coefficients themselves to be functions of the same coordinates that are being transformed— a condition that would then make the transformation non-linear.

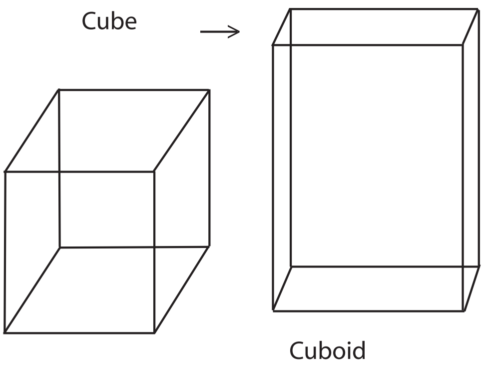

FIG. 2: M-H Transform

If the edges of the cube are not aligned with the coordinate axes, say the diagonal of a face falls on one of the axes instead, then the cube transforms into a parallelepiped. Regardless of the orientation of the shape, however, parallel lines remain parallel. The space is squeezed and stretched, but not pinched. Distortions remain strictly along axes lines, and for this reason we refer to these types of shape shifts as cuboid scale changes. Under such scale changes properly aligned cuboids remain cuboids for all possible transformation operators constructed from M-H numbers, when the coefficients of these numbers are independent of the coordinate variables undergoing transformation.

However, there's one additional complication to this scaling. From equation (2.82) we see that the M-H type of scaling permits scale factors with positive or negative sign. There is no restriction on the signs for the parameters, a_0, a_1, a_2, a_3. This means that the scaling may incorporate inversions of coordinate values, or reflections in planes through the origin of coordinates, along with the pure scaling operation. This type of transformation is therefore better described as a "`generalized nonproportional scaling`," because it may include "`inversions`."

INVERSIONS. In the xy-plane, an inversion of the x-coordinate value, i.e.

$$I_X : x \mapsto x' = -x, \tag{2.115}$$

is the same as a reflection in the y-axis, and is again the same as a 180° rotation about the y-axis, if the plane is allowed to rotate using the higher dimensional 3-space in which it is considered embedded. But, in 3-dimensional xyz-space, an inversion of the x-coordinate value cannot be made equal to any rotation. A 180° rotation about the y-axis would not only take the x-coordinate to its inverse position, $x \mapsto -x$, but would simultaneously take the z-coordinate value to its inverse position also, $z \mapsto -z$. An inversion of one coordinate value, through the origin, is still equal to a reflection. However, this time the mirror is the perpendicular *plane* passing through the origin. If the inversion changes the signs of two coordinate values simultaneously, i.e.

$$I_{XY} : (x, y) \mapsto (-x, -y), \tag{2.116}$$

then it can be made equal to a rotation, but not to any single reflection through a plane. And if the inversion involves all three coordinates of the 3-space, i.e.,

$$I_{XYZ} : (x, y, z) \mapsto (-x, -y, -z), \tag{2.117}$$

then there is neither rotation nor reflection that can be made to equal this transformation. Finally, the middle-hand M-H `hexpe number` may invert four coordinates all at the same time,

$$I_{WXYZ} : (w, x, y, z) \mapsto (-w, -x, -y, -z), \tag{2.118}$$

and again, neither a rotation nor a reflection can be constructed to produce the same result. Thus, apart from the one case given in (2.116), the M-H number never generates a rotation. The rotations are all performed by the R-H and L-H quaternions, while the M-H number generates a generalized scaling. What about the A-Z numbers?

The A-HAND inverse normalizing factor, N_A^4, is similar to the M-HAND factor, N_M^4, in (2.80), except the correspondence with the $[a_{ij}]$ matrix components in (2.35) is very different from that presented in (2.71),

$$a^2 - b^2 = (a-b)(a+b) \tag{2.119}$$

$$\begin{aligned} &= \left[(h_0^2 + h_{A1}^2 - h_{A2}^2 - h_{A3}^2) - (2h_0 h_{A1} + 2h_{A2}h_{A3})\right] \\ &\times \left[(h_0^2 + h_{A1}^2 - h_{A2}^2 - h_{A3}^2) + (2h_0 h_{A1} + 2h_{A2}h_{A3})\right] \end{aligned}$$

$$\begin{aligned} &= [h_0 - h_{A1} - h_{A2} - h_{A3}] \\ &\times [h_0 - h_{A1} + h_{A2} + h_{A3}] \\ &\times [h_0 + h_{A1} - h_{A2} + h_{A3}] \\ &\times [h_0 + h_{A1} + h_{A2} - h_{A3}] \end{aligned}$$

$$= \alpha_0\alpha_1\alpha_2\alpha_3 \tag{2.120}$$

$$4\alpha_0 = \begin{aligned} &+a_{00} - a_{01} - a_{02} - a_{03} \\ &-a_{10} + a_{11} + a_{12} + a_{13} \\ &-a_{20} + a_{21} + a_{22} + a_{23} \\ &-a_{30} + a_{31} + a_{32} + a_{33} \end{aligned} \qquad 4\alpha_1 = \begin{aligned} &+a_{00} - a_{01} + a_{02} + a_{03} \\ &-a_{10} + a_{11} - a_{12} - a_{13} \\ &+a_{20} - a_{21} + a_{22} + a_{23} \\ &+a_{30} - a_{31} + a_{32} + a_{33} \end{aligned}$$

$$4\alpha_2 = \begin{aligned} &+a_{00} - a_{01} + a_{02} + a_{03} \\ &-a_{10} + a_{11} - a_{12} - a_{13} \\ &+a_{20} - a_{21} + a_{22} + a_{23} \\ &+a_{30} - a_{31} + a_{32} + a_{33} \end{aligned} \qquad 4\alpha_3 = \begin{aligned} &+a_{00} + a_{01} + a_{02} - a_{03} \\ &+a_{10} + a_{11} + a_{12} - a_{13} \\ &+a_{20} + a_{21} + a_{22} - a_{23} \\ &-a_{30} - a_{31} - a_{32} + a_{33} \end{aligned}$$

Here the normalizing factor is the product of four parameters, $\alpha_0\alpha_1\alpha_2\alpha_3$, which are no longer simply the product of diagonal elements, instead each parameter is $\frac{1}{4}$ of the sum of all the components of the matrix $[a_{ij}]$ with 6 sign changes in various places in each sum.

Let us say that our A-H number is the usual,

$$h = h_0\boldsymbol{E} + h_{A1}\boldsymbol{I}_A + h_{A2}\boldsymbol{J}_A + h_{A3}\boldsymbol{K}_A \tag{2.121}$$

which is also, therefore,

$$h = \begin{pmatrix} h_0 & h_{A1} & h_{A2} & h_{A3} \\ h_{A1} & h_0 & -h_{A3} & -h_{A2} \\ h_{A2} & -h_{A3} & h_0 & -h_{A1} \\ h_{A3} & -h_{A2} & -h_{A1} & h_0 \end{pmatrix} \tag{2.122}$$

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} h_0 w + h_{A1}x + h_{A2}y + h_{A3}z \\ h_{A1}w + h_0 x - h_{A3}y - h_{A2}z \\ h_{A2}w - h_{A3}x + h_0 y - h_{A1}z \\ h_{A3}w - h_{A2}x - h_{A1}y + h_0 z \end{pmatrix}$$

When we apply this transformation operator, h, to our quaternion 4-space point, $q = (w, x, y, z)$, this transforms the coordinates into the new point, $q' = (w', x', y', z')$.

The Z-HAND inverse normalizing factor, N_Z^4, is similar to the M-HAND factor, N_M^4, in (2.80), except the correspondence with the $[a_{ij}]$ matrix components in (2.35) is very different from that presented in (2.71),

$$a^2 - b^2 = (a-b)(a+b) \tag{2.123}$$

$$\begin{aligned} &= \left[(h_0^2 + h_{Z1}^2 - h_{Z2}^2 - h_{Z3}^2) - (2h_0 h_{Z1} + 2h_{Z2}h_{Z3})\right] \\ &\times \left[(h_0^2 + h_{Z1}^2 - h_{Z2}^2 - h_{Z3}^2) + (2h_0 h_{Z1} + 2h_{Z2}h_{Z3})\right] \end{aligned}$$

$$\begin{aligned} &= [h_0 - h_{Z1} - h_{Z2} - h_{Z3}] \\ &\times [h_0 - h_{Z1} + h_{Z2} + h_{Z3}] \\ &\times [h_0 + h_{Z1} - h_{Z2} + h_{Z3}] \\ &\times [h_0 + h_{Z1} + h_{Z2} - h_{Z3}] \end{aligned}$$

$$= \alpha_0\alpha_1\alpha_2\alpha_3 \tag{2.124}$$

$$4\alpha_0 = \begin{aligned} &+a_{00} + a_{01} + a_{02} + a_{03} \\ &+a_{10} + a_{11} + a_{12} + a_{13} \\ &+a_{20} + a_{21} + a_{22} + a_{23} \\ &+a_{30} + a_{31} + a_{32} + a_{33} \end{aligned} \qquad 4\alpha_1 = \begin{aligned} &+a_{00} + a_{01} - a_{02} - a_{03} \\ &+a_{10} + a_{11} - a_{12} - a_{13} \\ &-a_{20} - a_{21} + a_{22} + a_{23} \\ &-a_{30} - a_{31} + a_{32} + a_{33} \end{aligned}$$

$$4\alpha_2 = \begin{aligned} &+a_{00} - a_{01} + a_{02} - a_{03} \\ &-a_{10} + a_{11} - a_{12} + a_{13} \\ &+a_{20} - a_{21} + a_{22} - a_{23} \\ &-a_{30} + a_{31} - a_{32} + a_{33} \end{aligned} \qquad 4\alpha_3 = \begin{aligned} &+a_{00} - a_{01} - a_{02} + a_{03} \\ &-a_{10} + a_{11} + a_{12} - a_{13} \\ &-a_{20} + a_{21} + a_{22} - a_{23} \\ &+a_{30} - a_{31} - a_{32} + a_{33} \end{aligned}$$

Here the normalizing factor is the product of four parameters, $\alpha_0\alpha_1\alpha_2\alpha_3$, which are no longer simply the product of diagonal elements, instead each parameter is $\frac{1}{4}$ of the sum of all the components of the matrix $[a_{ij}]$ with 8 sign changes in three of the sums.

Let us say that our Z-H number is the usual,

$$h = h_0\boldsymbol{E} + h_{Z1}\boldsymbol{I}_Z + h_{Z2}\boldsymbol{J}_Z + h_{Z3}\boldsymbol{K}_Z \tag{2.125}$$

which is also, therefore,

$$h = \begin{pmatrix} h_0 & -h_{Z1} & -h_{Z2} & -h_{Z3} \\ -h_{Z1} & h_0 & -h_{Z3} & -h_{Z2} \\ -h_{Z2} & -h_{Z3} & h_0 & -h_{Z1} \\ -h_{Z3} & -h_{Z2} & -h_{Z1} & h_0 \end{pmatrix} \tag{2.126}$$

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} h_0 w - h_{Z1}x - h_{Z2}y - h_{Z3}z \\ -h_{Z1}w + h_0 x - h_{Z3}y - h_{Z2}z \\ -h_{Z2}w - h_{Z3}x + h_0 y - h_{Z1}z \\ -h_{Z3}w - h_{Z2}x - h_{Z1}y + h_0 z \end{pmatrix}$$

When we apply this transformation operator, h, to our quaternion 4-space point, $q = (w, x, y, z)$, this transforms the coordinates into the new point, $q' = (w', x', y', z')$.

Invariant Quartic Forms. Like the M-H numbers, neither the A-H nor the Z-H preserve distances. Unlike the M-H numbers, neither the A-H nor the Z-H preserve 4-volumes. Instead, the A-H preserves the quantity measured by the same `quartic norm` expression in (2.80).

$$N_A^4(w,x,y,z) \equiv (+w-x-y-z) \times (+w-x+y+z) \times (+w+x-y+z) \times (+w+x+y-z) \tag{2.127}$$

$$\equiv N_Z^4(w,x,y,z) \tag{2.128}$$

To see this, note that under the A-H transformation presented in (2.122), the coordinates (w,x,y,z) become (w',x',y',z'), and the following four identities hold,

$$\begin{aligned} w'-x'-y'-z' &= (h_0-h_{A1}-h_{A2}-h_{A3}) \times (w-x-y-z) \\ w'-x'+y'+z' &= (h_0-h_{A1}+h_{A2}+h_{A3}) \times (w-x+y+z) \\ w'+x'-y'+z' &= (h_0+h_{A1}-h_{A2}+h_{A3}) \times (w+x-y+z) \\ w'+x'+y'-z' &= (h_0+h_{A1}+h_{A2}-h_{A3}) \times (w+x+y-z) \end{aligned} \tag{2.129}$$

therefore we have the identity,

$$\begin{aligned} N_A^4(w',x',y',z') &= N_A^4(h_0,h_{A1},h_{A2},h_{A3}) \cdot N_A^4(w,x,y,z) \\ &= \alpha_0\alpha_1\alpha_2\alpha_3 \cdot N_A^4(w,x,y,z) \end{aligned} \tag{2.130}$$

So, now, when the normalizing factor is 1, i.e. $N_A^4(h_0,h_{A1},h_{A2},h_{A3}) = \alpha_0\alpha_1\alpha_2\alpha_3 = 1$, then the form of the `quartic norm` is an invariant quantity for the transforming coordinates, $N_A^4(w',x',y',z') = N_A^4(w,x,y,z)$.

SQUARE NORM. Perhaps the most important property of the R-H `square norm` given in equation (2.79) is the "`four squares rule`", which says that *the product of two sums of four squares is the sum of four squares.* If we're given two R-H quaternions, h and g, and we the let, $N_R^2(h) \equiv N_R^2(h_0,h_{R1},h_{R2},h_{R3})$, we can write this rule,

$$N_R^2(h)N_R^2(g) = N_R^2(hg) \tag{2.131}$$

From the results above, we now see that a somewhat similar rule holds for the A-H `quartic norm`. If we're given two A-H numbers, h and g, and we similarly let, $N_A^4(h) \equiv N_A^4(h_0,h_{A1},h_{A2},h_{A3})$, we can write the corresponding quartic rule for the product of four sums,

$$N_A^4(h)N_A^4(g) = N_A^4(hg) \tag{2.132}$$

Note that the R-H `square norm` and the A-H `quartic norm` are each the actual normalizing factors for construction of the inverses for the respective 4-d hypercomplex numbers, AND at the same time, are the invariant forms relevant for the transforming coordinates. The same cannot be said for the M-H and Z-H numbers, but the L-H numbers are similar in this respect to the R-H.

Now, the Z-H numbers do not preserve a coordinate space quantity measured by this `quartic norm` expression, but rather a different complementary fourth power form is held invariant instead. Consider the following alternative `quartic form`,

$$Q_Z^4(w,x,y,z) \equiv (+w+x+y+z) \times (-w-x+y+z) \times (-w+x-y+z) \times (-w+x+y-z) \tag{2.133}$$

Under the Z-H transformation presented in (2.126), the coordinates (w,x,y,z) become (w',x',y',z'), and the following identities hold,

$$\begin{aligned} +w'+x'+y'+z' &= (h_0-h_{Z1}-h_{Z2}-h_{Z3}) \times (+w+x+y+z) \\ -w'-x'+y'+z' &= (h_0-h_{Z1}+h_{Z2}+h_{Z3}) \times (-w-x+y+z) \\ -w'+x'-y'+z' &= (h_0+h_{Z1}-h_{Z2}+h_{Z3}) \times (-w+x-y+z) \\ -w'+x'+y'-z' &= (h_0+h_{Z1}+h_{Z2}-h_{Z3}) \times (-w+x+y-z) \end{aligned} \tag{2.134}$$

therefore we have the identity,

$$\begin{aligned} Q_Z^4(w',x',y',z') &= N_Z^4(h_0,h_{Z1},h_{Z2},h_{Z3}) \cdot Q_Z^4(w,x,y,z) \\ &= \alpha_0\alpha_1\alpha_2\alpha_3 \cdot Q_Z^4(w,x,y,z) \end{aligned} \tag{2.135}$$

So, now, when the normalizing factor is 1, i.e. $N_Z^4(h_0,h_{Z1},h_{Z2},h_{Z3}) = \alpha_0\alpha_1\alpha_2\alpha_3 = 1$, then it is the alternative `quartic form`, Q_Z^4, that is an invariant quantity, this time, $Q_Z^4(w',x',y',z') = Q_Z^4(w,x,y,z)$.

QUARTIC NORM. From eq (2.135), by applying two Z-H transformations in succession, we can demonstrate that a rule of the form (2.132) holds for these numbers, when, for a given Z-H number, h, we let, $N_Z^4(h) \equiv N_Z^4(h_0,h_{Z1},h_{Z2},h_{Z3})$. And again, it's trivial to see from (2.83) that the M-H numbers also obey this rule, when, for a given M-H number, h, we let, $N_M^4(h) \equiv N_M^4(h_0,h_{M1},h_{M2},h_{M3})$. We may state this rule of the quartic norm thus—*The product of two products of four sums is the product four sums with the same form*—it being understood that the *form* is that specified in the definition of the norm. Now let's define,

$$Q_M^4 = wxyz \tag{2.136}$$

$$Q_A^4 = N_A^4(w,x,y,z) \tag{2.137}$$

$$Q_Z^4 = Q_Z^4(w,x,y,z) \tag{2.138}$$

$$S_R^2 = N_R^2(w,x,y,z) \tag{2.139}$$

$$S_L^2 = N_L^2(w,x,y,z) \tag{2.140}$$

Given that the form of the norm is not always the same as the form of some coordinate space measure which turns out to be invariant under the transformation, we give the invariant quantities their own symbols, $Q^4_M, Q^4_A, Q^4_Z, S^2_R, S^2_L$, defined by these expressions above.

It may be interesting to note that the R-L basis elements are of order 4, i.e. they have, $u^4 = 1$, and here we have an invariant quantity which is a polynomial of order 2, while the M-A-Z basis elements are of order 2, i.e. they have, $u^2 = 1$, and in this case we have invariant quantities which are polynomials of order 4.

Image Volumes. While the A-H and Z-H transformations do not preserve the 4-volume of the object, like the M-H numbers do, they nevertheless preserve the 4-volume of a special image of the object. Consider, for example, the coordinate transformation,

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} +1 & -1 & -1 & -1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} +w-x-y-z \\ +w-x+y+z \\ +w+x-y+z \\ +w+x+y-z \end{pmatrix} \tag{2.141}$$

This transformation sets up image points in a new coordinate space, corresponding to points in our original space. An object in our space will have an image in this corresponding space. Whenever the object is transformed in our space, $(w, x, y, z) \mapsto (w', x', y', z')$, its image will also be transformed in the corresponding space, $(W, X, Y, Z) \mapsto (W', X', Y', Z')$. When we apply the A-H number transform, h, to the coordinates, (w, x, y, z), the 4-volume in the new space becomes,

$$W'X'Y'Z' = N^4_A(h) \cdot WXYZ \tag{2.142}$$

so that when the A-H quartic norm is 1, i.e. $N^4_A(h) = \alpha_0\alpha_1\alpha_2\alpha_3 = 1$, the 4-volume of this image is unchanged by the transformation. The image undergoes the type of cuboid shape shift as if being transformed by an M-H number instead.

A similar situation holds for the Z-H numbers. Consider the alternative coordinate transformation,

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} +w+x+y+z \\ -w-x+y+z \\ -w+x-y+z \\ -w+x+y-z \end{pmatrix} \tag{2.143}$$

This transformation also sets up image points in another new coordinate space, corresponding to points in our original space. An object in our space will also have a second image therefore in this new space. Whenever the object is transformed in our space, $(w, x, y, z) \mapsto (w', x', y', z')$, its second image will also be transformed in the corresponding space, $(W, X, Y, Z) \mapsto (W', X', Y', Z')$. When we apply the Z-H number transform, h, to the coordinates, (w, x, y, z), the 4-volume in the new alternate space becomes,

$$W'X'Y'Z' = N^4_Z(h) \cdot WXYZ \tag{2.144}$$

Now, when the Z-H quartic norm is 1, i.e. $N^4_Z(h) = \alpha_0\alpha_1\alpha_2\alpha_3 = 1$, the 4-volume of this image is unchanged by the transformation. The image once more undergoes the type of cuboid shape shift as if being transformed by an M-H number instead.

The significance of the invariants, Q^4_A and Q^4_Z, is now apparent. The existence of such invariant forms implicate the existence of parallel corresponding spaces where the transforms can be described in simpler terms—`non-proportional scale` changes—there being a fixed transform connecting the object space with the image spaces where descriptions are more easily understood. The contorted shape shifting of the object in its own space, involving reflections, inversions, and scale changes, is then dissected into its component parts, separating its rather more complicated inverting profile from its size distortions, so that the scale change can be presented by itself. Then we discover that this non-proportional scale change is the only variable in the transformation. The other component parts of the transformation are fixed and don't depend on the coefficients of the transformation operator at all.

These two image spaces are themselves obtained via matrices which are actually inverses of each other. This is easily seen by multiplying the two coordinate transform matrices together.

$$\begin{pmatrix} +1 & -1 & -1 & -1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 \end{pmatrix} \begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \tag{2.145}$$

The fixed coordinate transform in the first case is practically the inverse of the fixed coordinate transform in the second case, except for an overall scale factor of 4. We can therefore define a transform matrix with an extra normalizing factor of 1/2 to make these exact inverses of each other. Let's call the new matrix, T.

$$\boldsymbol{T} = \frac{1}{2}\begin{pmatrix} +1 & -1 & -1 & -1 \\ +1 & -1 & +1 & +1 \\ +1 & +1 & -1 & +1 \\ +1 & +1 & +1 & -1 \end{pmatrix}, \quad \boldsymbol{T}^{-1} = \frac{1}{2}\begin{pmatrix} +1 & +1 & +1 & +1 \\ -1 & -1 & +1 & +1 \\ -1 & +1 & -1 & +1 \\ -1 & +1 & +1 & -1 \end{pmatrix} \tag{2.146}$$

Now we re-define the coordinate transformations to those two image spaces, and except for the extra 1/2 that appears in the formulas the results are basically the same—the 4-volumes in the image spaces are still invariant when the quartic norms are 1. Only now, we have,

A-H IMAGE SPACE.

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \boldsymbol{T} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} +w-x-y-z \\ +w-x+y+z \\ +w+x-y+z \\ +w+x+y-z \end{pmatrix} \qquad (2.147)$$

Z-H IMAGE SPACE.

$$\begin{pmatrix} W \\ X \\ Y \\ Z \end{pmatrix} = \boldsymbol{T}^{-1} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} +w+x+y+z \\ -w-x+y+z \\ -w+x-y+z \\ -w+x+y-z \end{pmatrix} \qquad (2.148)$$

Now, for the A-H case, $WXYZ = \frac{1}{16} \cdot Q_A^4(w,x,y,z)$, and $WXYZ = \frac{1}{16} \cdot Q_Z^4(w,x,y,z)$, for the Z-H case. The 4-volumes are reduced by a factor of 1/16, but are otherwise the same.

Given that `inverse transformations`, $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$, map the coordinates to these two image spaces, the image of the object in each of these image spaces are in some sense `reverse images` of each other. If the image of the object skews to the left in one image space, it skews to the right in the other image space instead.

A-H CASE. If we transform (w,x,y,z) into the the image (W,X,Y,Z), then multiply by a M-H number, then transform the (W',X',Y',Z') back into (w',x',y',z') of our coordinate space, we'll have exactly the same effect as multiplying by an A-H number in the object's space.

$$h_A = \boldsymbol{T}^{-1} h_M \boldsymbol{T} \qquad (2.149)$$

$$\begin{aligned} h_M &= h_0\boldsymbol{E} + h_1\boldsymbol{I}_M + h_2\boldsymbol{J}_M + h_3\boldsymbol{K}_M \\ h_A &= h_0\boldsymbol{E} + h_1\boldsymbol{I}_A + h_2\boldsymbol{J}_A + h_3\boldsymbol{K}_A \end{aligned}$$

This basically means that the four "variable" coefficients of the A-H number, $\{h_0, h_1, h_2, h_3\}$, are all parameters that describe the `non-proportional scaling` component of the shape shift, while the four unchanging basis elements, $\{\boldsymbol{E}, \boldsymbol{I}_A, \boldsymbol{J}_A, \boldsymbol{K}_A\}$, are all that's left to describe the other components of the transformation, like reflections, inversions, shears and skews. Therefore, these latter type of changes to the object comprise a "fixed" transformation, independent of these variable coefficients, that holds for all transformations of the A-H type, and that can then be separated by single fixed transform, of the type, $\boldsymbol{T}$, to reveal the true effect of different A-H numbers on the object's coordinates. One A-H number is differentiated from another A-H number, solely by the scaling transform of M-H type hidden within its rather more complex representation.

Z-H CASE. Again, if we transform (w,x,y,z) into the the image (W,X,Y,Z), then multiply by a M-H number, then transform the (W',X',Y',Z') back into (w',x',y',z') of our coordinate space, we'll have exactly the same effect as multiplying by an Z-H number in the object's space.

$$h_Z = \boldsymbol{T} h_M \boldsymbol{T}^{-1} \qquad (2.150)$$

$$\begin{aligned} h_M &= h_0\boldsymbol{E} + h_1\boldsymbol{I}_M + h_2\boldsymbol{J}_M + h_3\boldsymbol{K}_M \\ h_Z &= h_0\boldsymbol{E} + h_1\boldsymbol{I}_Z + h_2\boldsymbol{J}_Z + h_3\boldsymbol{K}_Z \end{aligned}$$

The situation is analogous to the A-H transform. Only the scale change distinguishes one Z-H number from another. The Z-H generates a variable non-proportional scale change together with a fixed component made up of reflections, inversions, shears and skews. But, essentially, it disguises a cuboid type shape shift that is really identical to that produced by the M-H numbers.

These three M-A-Z middle-hand numbers then, not only have cuboid type weight factors for the construction of their inverses, they all also generate shape shifting transformations that are essentially cuboid type scale changes at the core.

X-HANDS. The M-A-Z numbers are not the only commutative 4-d hypercomplex sub-algebras of the `hexpe system`. If we cross these hands we obtain three more similar algebras. Say we pick any one of the the three sets of four basis elements, $\{\boldsymbol{E}, \boldsymbol{I}_M, \boldsymbol{I}_A, \boldsymbol{I}_Z\}$, $\{\boldsymbol{E}, \boldsymbol{J}_M, \boldsymbol{J}_A, \boldsymbol{J}_Z\}$, $\{\boldsymbol{E}, \boldsymbol{K}_M, \boldsymbol{K}_A, \boldsymbol{K}_Z\}$, and re-label the four elements, $\{E, I, J, K\}$, respectively. Then we'd have another commutative hypercomplex number with the product rules,

$$\begin{aligned} E^2 &= +E, \\ I^2 &= J^2 = K^2 = E, \quad IJ = JI = K, \quad (2.151) \\ JK &= KJ = I, \quad KI = IK = J. \end{aligned}$$

X1-H. Consider then, the following number,

$$h = h_0\boldsymbol{E} + h_{M1}\boldsymbol{I}_M + h_{A1}\boldsymbol{I}_A + h_{Z1}\boldsymbol{I}_Z \qquad (2.152)$$

which is also, therefore,

$$h = \begin{pmatrix} h_0 - h_{M1} & h_{A1} - h_{Z1} & 0 & 0 \\ h_{A1} - h_{Z1} & h_0 - h_{M1} & 0 & 0 \\ 0 & 0 & h_0 + h_{M1} & -h_{A1} - h_{Z1} \\ 0 & 0 & -h_{A1} - h_{Z1} & h_0 + h_{M1} \end{pmatrix}$$

When we apply this transformation operator, h, to our quaternion 4-space point, $q = (w, x, y, z)$, this transforms the coordinates into the new point, $q' = (w', x', y', z')$.

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (h_0 - h_{M1})w + (h_{A1} - h_{Z1})x \\ (h_{A1} - h_{Z1})w + (h_0 - h_{M1})x \\ (h_0 + h_{M1})y - (h_{A1} + h_{Z1})z \\ -(h_{A1} + h_{Z1})y + (h_0 + h_{M1})z \end{pmatrix}$$

and the coordinates now have an invariant quantity given by the following quartic form,

$$\begin{aligned} Q^4_{X1}(w, x, y, z) \equiv\ & (+w + x) \\ & \times (+w - x) \\ & \times (+y + z) \\ & \times (+y - z) \\ \\ \equiv\ & (w^2 - x^2)(y^2 - z^2) \end{aligned} \tag{2.153}$$

We can see this by verifying that when the coordinates (w, x, y, z) become (w', x', y', z'), the following four identities hold,

$$\begin{aligned} w' + x' &= (h_0 - h_{M1} + h_{A1} - h_{Z1}) \\ &\quad \times (w + x) \\ w' - x' &= (h_0 - h_{M1} - h_{A1} + h_{Z1}) \\ &\quad \times (w - x) \\ y' + z' &= (h_0 + h_{M1} - h_{A1} - h_{Z1}) \\ &\quad \times (y + z) \\ y' - z' &= (h_0 + h_{M1} + h_{A1} + h_{Z1}) \\ &\quad \times (y - z) \end{aligned} \tag{2.154}$$

therefore we have the identity,

$$\begin{aligned} Q^4_{X1}(w', x', y', z') &= N^4_{X1}(h_0, h_{M1}, h_{A1}, h_{Z1}) \cdot Q^4_{X1}(w, x, y, z) \\ &= \alpha_0\alpha_1\alpha_2\alpha_3 \cdot Q^4_{X1}(w, x, y, z) \end{aligned} \tag{2.155}$$

Where the normalizing factor, N^4_{X1}, has the definition,

$$\begin{aligned} N^4_{X1}(w, x, y, z) \equiv\ & (+w + x + y + z) \\ & \times (-w - x + y + z) \\ & \times (-w + x - y + z) \\ & \times (-w + x + y - z) \\ & \times -1 \end{aligned} \tag{2.156}$$

and except for the overall extra minus sign, -1, this is the same form as Q^4_Z, given in (2.133). Now to see that this is indeed the correct normalizing factor, lets compute the inverse of this `hexpe number`. We start with the h number in (2.152), flip a pair of signs, and define a new number,

$$g = h_0\boldsymbol{E} + h_{M1}\boldsymbol{I}_M - h_{A1}\boldsymbol{I}_A - h_{Z1}\boldsymbol{I}_Z \tag{2.157}$$

then take the product, gh,

$$\begin{aligned} gh &= (h_0^2 + h_{M1}^2 - h_{A1}^2 - h_{Z1}^2)\boldsymbol{E} \\ &+ (2h_0h_{M1} - 2h_{A1}h_{Z1})\boldsymbol{I}_M \end{aligned} \tag{2.158}$$

Now define the usual, $f = (a\boldsymbol{E} - b\boldsymbol{I}_M)$, complementary factor,

$$\begin{aligned} f &= (h_0^2 + h_{M1}^2 - h_{A1}^2 - h_{Z1}^2)\boldsymbol{E} \\ &- (2h_0h_{M1} - 2h_{A1}h_{Z1})\boldsymbol{I}_M \end{aligned} \tag{2.159}$$

and take the product, fgh,

$$fgh = (a^2 - b^2)\boldsymbol{E} \tag{2.160}$$

`where`

$$a = (h_0^2 + h_{M1}^2 - h_{A1}^2 - h_{Z1}^2) \tag{2.161}$$

$$b = (2h_0h_{M1} - 2h_{A1}h_{Z1}) \tag{2.162}$$

This result is similar to that obtained in (2.50-52), except we have, $-2h_{A1}h_{Z1}$, here in the new expression for b, where the previous result was, $+2h_{M2}h_{M3}$. This occurs because the form of our commutative rule is now, $JK = +I$, instead of, $JK = -I$, etc., so there are subtle sign changes to our new expressions.

Our inverse, h^{-1} is given by the usual product, fg, divided by this normalizing factor, $(a^2 - b^2)$,

$$h^{-1} = \frac{w_0\boldsymbol{E} + w_1\boldsymbol{I}_{M1} + w_2\boldsymbol{I}_{A1} + w_3\boldsymbol{I}_{Z1}}{a^2 - b^2} \tag{2.163}$$

where

$$\begin{aligned} a^2 - b^2 &= h_0^4 + h_{M1}^4 + h_{A1}^4 + h_{Z1}^4 \\ &- 2h_0^2h_{M1}^2 - 2h_0^2h_{A1}^2 - 2h_0^2h_{Z1}^2 \\ &- 2h_{A1}^2h_{Z1}^2 - 2h_{M1}^2h_{Z1}^2 - 2h_{M1}^2h_{A1}^2 \\ &+ 8h_0h_{M1}h_{A1}h_{Z1} \end{aligned} \tag{2.164}$$

`and`,

$$\begin{aligned} w_0 &= h_0^3 - h_0(h_{M1}^2 + h_{A1}^2 + h_{Z1}^2) + 2h_{M1}h_{A1}h_{Z1} \\ w_1 &= h_{M1}^3 - h_{M1}(h_0^2 + h_{A1}^2 + h_{Z1}^2) + 2h_0h_{A1}h_{Z1} \\ w_2 &= h_{A1}^3 - h_{A1}(h_{M1}^2 + h_0^2 + h_{Z1}^2) + 2h_{M1}h_0h_{Z1} \\ w_3 &= h_{Z1}^3 - h_{Z1}(h_{M1}^2 + h_{A1}^2 + h_0^2) + 2h_{M1}h_{A1}h_0 \end{aligned}$$

The normalizing factor can also be written,

$$a^2 - b^2 = (a - b)(a + b) \tag{2.165}$$

$$\begin{aligned} &= [(h_0^2 + h_{M1}^2 - h_{A1}^2 - h_{Z1}^2) - (2h_0h_{M1} - 2h_{A1}h_{Z1})] \\ &\times [(h_0^2 + h_{M1}^2 - h_{A1}^2 - h_{Z1}^2) + (2h_0h_{M1} - 2h_{A1}h_{Z1})] \end{aligned}$$

$$\begin{aligned} &= [h_0 - h_{M1} - h_{A1} + h_{Z1}] \\ &\times [h_0 - h_{M1} + h_{A1} - h_{Z1}] \\ &\times [h_0 + h_{M1} - h_{A1} - h_{Z1}] \\ &\times [h_0 + h_{M1} + h_{A1} + h_{Z1}] \end{aligned}$$

$$= \alpha_0\alpha_1\alpha_2\alpha_3 \tag{2.166}$$

A similar analysis can be done for the X2-H defined by $\{\boldsymbol{E}, \boldsymbol{J}_M, \boldsymbol{J}_A, \boldsymbol{J}_Z\}$, and the X3-H defined by $\{\boldsymbol{E}, \boldsymbol{K}_M, \boldsymbol{K}_A, \boldsymbol{K}_Z\}$. These X-HANDS are all cuboid scaling like the pure M-A-Z numbers, and by inverting one or three imaginary axes we can show they are in fact isomorphic to the middle-hand numbers given in (2.32).

The General Inverse. Now that we've seen a few simple special case examples, let's return to the multiplicative inverse of the most general `hexpe` number of the form, h, given in (2.34). Let this inverse be, $h^{-1} = \sum_P h'_P \boldsymbol{E}_P$, where, $P \in \{0, R1, R2, ..., A1, ..., Z3\}$, so,

$$\begin{aligned} h^{-1} = h'_0\mathbf{E} & \\ + h'_{R1}\mathbf{I_R} + h'_{R2}\mathbf{J_R} + h'_{R3}\mathbf{K_R} & \\ + h'_{L1}\mathbf{I_L} + h'_{L2}\mathbf{J_L} + h'_{L3}\mathbf{K_L} & \\ + h'_{M1}\mathbf{I_M} + h'_{M2}\mathbf{J_M} + h'_{M3}\mathbf{K_M} & \\ + h'_{A1}\mathbf{I_A} + h'_{A2}\mathbf{J_A} + h'_{A3}\mathbf{K_A} & \\ + h'_{Z1}\mathbf{I_Z} + h'_{Z2}\mathbf{J_Z} + h'_{Z3}\mathbf{K_Z} & \end{aligned} \tag{2.167}$$

Now, if we write the `hexpe number`, h, in the matrix form $[a_{ij}]$, then invert this matrix to get, $[a_{ij}]^{-1} = [F_{ji}]/d$, where, $d = \mathtt{det}([a_{ij}])$, then we can write h^{-1}, in the form,

$$\begin{aligned} h^{-1} = (w_0\mathbf{E} & \\ + w_{R1}\mathbf{I_R} + w_{R2}\mathbf{J_R} + w_{R3}\mathbf{K_R} & \\ + w_{L1}\mathbf{I_L} + w_{L2}\mathbf{J_L} + w_{L3}\mathbf{K_L} & \\ + w_{M1}\mathbf{I_M} + w_{M2}\mathbf{J_M} + w_{M3}\mathbf{K_M} & \\ + w_{A1}\mathbf{I_A} + w_{A2}\mathbf{J_A} + w_{A3}\mathbf{K_A} & \\ + w_{Z1}\mathbf{I_Z} + w_{Z2}\mathbf{J_Z} + w_{Z3}\mathbf{K_Z})/d & \end{aligned} \tag{2.168}$$

where we've expressed the h'_P coefficients in terms of weight factors divided by the determinant, $h'_P = w_P/d$. This determinant, d, is now our normalizing factor.

The w_P weight factors are constructed from the cofactor components, F_{ij}, which are themselves determined from the original matrix components, a_{ij}, by taking the Minors with alternating signs, $F_{ij} = (-1)^{i+j} M_{ij}$. Thus we simply take the equations (2.37), replace the "a" letters with "F" and remember to swap the the digits in the index subscripts, because we need to transpose the cofactor to use its components here. Next, replace the "F" with "M", remembering to change the signs wherever the index pair has an odd sum, then replace the Minors with their a-component expressions. Finally, we replace the a-components with the h-coefficient expressions in (2.36) of the original hexpe number.

Our weight factors are then expressed in terms of the original hexpe coefficients,

$$w_P = w_P(h_0, h_{R1}, \ldots, h_{Z3}) \tag{2.169}$$

The results and these intermediate substitutions are illustrated in (TABLE `T.3`).

The weight factors, w_P, all have the familiar cubic form we met before. Each factor is the sum of the usual cube, square cuboid, and general cuboid expressions. But, this time the square cuboid has 15 squares instead of 3, and the general cuboid has a corresponding 15 terms, so that a total of 31 terms make up the summation expression for each weight factor. Nevertheless, the form is familiar.

.	h^2_{M1}	h^2_{M2}	h^2_{M3}	h^2_{R1}	h^2_{R2}	h^2_{R3}	h^2_{L1}	h^2_{L2}	h^2_{L3}	h^2_{A1}	h^2_{A2}	h^2_{A3}	h^2_{Z1}	h^2_{Z2}	h^2_{Z3}
w_0	+	+	+	−	−	−	−	−	−	+	+	+	+	+	+
w_{M1}	+	+	+	−	+	+	−	+	+	+	−	−	+	−	−
w_{M2}	+	+	+	+	−	+	+	−	+	−	+	−	−	+	−
w_{M3}	+	+	+	+	+	−	+	+	−	−	−	+	−	−	+
w_{R1}	+	−	−	+	+	+	−	−	−	−	−	+	−	+	−
w_{R2}	−	+	−	+	+	+	−	−	−	+	−	−	−	−	+
w_{R3}	−	−	+	+	+	+	−	−	−	−	+	−	+	−	−
w_{L1}	+	−	−	−	−	−	+	+	+	−	+	−	−	−	−
w_{L2}	−	+	−	−	−	−	+	+	+	−	−	+	+	−	−
w_{L3}	−	−	+	−	−	−	+	+	+	+	−	−	−	+	−
w_{A1}	+	−	−	+	−	+	+	+	−	+	+	+	+	−	−
w_{A2}	−	+	−	+	+	−	−	+	+	+	+	+	−	+	−
w_{A3}	−	−	+	−	+	+	+	−	+	+	+	+	−	−	+
w_{Z1}	+	−	−	+	+	−	+	−	+	+	−	−	+	+	+
w_{Z2}	−	+	−	−	+	+	+	+	−	−	+	−	+	+	+
w_{Z3}	−	−	+	+	−	+	−	+	+	−	−	+	+	+	+

The signs $s_{P,\alpha}$ on the square terms.

$$w_P = h_P^3 - h_P \sum_{\alpha} s_{P,\alpha} h_\alpha^2 - 2 \sum_{\alpha\beta\gamma} s_{P,\alpha\beta\gamma} h_\alpha h_\beta h_\gamma$$

The first term is the cube of the corresponding, h_P, coefficient. Next come the sum of 15 terms that make up the square cuboid. The signs, $s_{P,\alpha}$, on these squares are easy to recall. First we notice that all the imaginary coefficients are present among these squares in the formula for w_0. The signs on the squares are positive for the M-A-Z order 2 elements ($e^2 = +1$), and negative for the R-L order 4 quaternions ($e^2 = -1$). Then, to determine the formula for another weight factor, w_P, we first modify the signs according to the table given above on the 15 elements, then substitute the h_P coefficient among these 15 squares with h_0. Note which signs change according to the chosen P. Finally, the third part of the expression consisting of 15 general cuboid terms are included. These results are presented in (TABLE `T.3-IV`).

The normalizing factor, d, which is the determinant of the 4×4 matrix form for the hexpe number, can be written in simple form in terms of the h-coefficients and these weight factors.

$$\begin{aligned} d &= h_0 \cdot w_0 \\ &+ h_{M1} \cdot w_{M1} + h_{M2} \cdot w_{M2} + h_{M3} \cdot w_{M3} \\ &+ h_{A1} \cdot w_{A1} + h_{A2} \cdot w_{A2} + h_{A3} \cdot w_{A3} \\ &+ h_{Z1} \cdot w_{Z1} + h_{Z2} \cdot w_{Z2} + h_{Z3} \cdot w_{Z3} \\ &- h_{R1} \cdot w_{R1} - h_{R2} \cdot w_{R2} - h_{R3} \cdot w_{R3} \\ &- h_{L1} \cdot w_{L1} - h_{L2} \cdot w_{L2} - h_{L3} \cdot w_{L3} \end{aligned} \tag{2.170}$$

$$\begin{aligned} &= h_0 \cdot w_0 - \sum_{k=1,2,3} (h_{Rk} \cdot w_{Rk} + h_{Lk} \cdot w_{Lk}) \\ &+ \sum_{k=1,2,3} (h_{Mk} \cdot w_{Mk} + h_{Ak} \cdot w_{Ak} + h_{Zk} \cdot w_{Zk}) \end{aligned}$$

Note again, the + signs on the M-A-Z, and - signs on the R-L. This suggests we may write,

$$d = \sum_P h_P w_P \boldsymbol{E}_P \cdot \boldsymbol{E}_P = \sum_P (h_P \boldsymbol{E}_P) \cdot (w_P \boldsymbol{E}_P)$$

or, even define a "dot product" for hexpe, $d = \boldsymbol{h} \circ \boldsymbol{w}$.

The Bilateral Factor. Let us review equation (2.29) again, which we'll write here in our newer basis labels and coefficient notation,

$$(h_0\boldsymbol{E} + h_{R1}\boldsymbol{I}_R + h_{R2}\boldsymbol{J}_R + h_{R3}\boldsymbol{K}_R + h_{L1}\boldsymbol{I}_L + h_{L2}\boldsymbol{J}_L + h_{L3}\boldsymbol{K}_L)q = C \tag{2.171}$$

We'll refer to this combined parameter with right and left terms in parenthesis as `the bilateral factor`. There are no terms from the M-A-Z numbers appearing here, just R-L coefficients. We can now use the general hexpe inverse formula, we just presented, to find the corresponding multiplicative inverse for this special number. When the scalar coefficient, h_0, is zero, the inverse will likewise only contain R-L terms, but otherwise the M-A-Z terms will also appear in the inverse.

Let us now use the letter, $\boldsymbol{A}$, to represent this bilateral factor, so $\boldsymbol{A}^{-1}$ will be our inverse. Our equation is of the form, $\boldsymbol{A}q = C$. We can therefore write, $q = \boldsymbol{A}^{-1}C$. The formula for this inverse is shown on the right. ⇒ Now, $\boldsymbol{A}^{-1}$ is an `hexpe number` in 4×4 matrix format, but, q and C are quaternions in 4×1 matrix format.

We would like to express these two quaternions, q, C, in 4×4 matrix format also, so that all the parameters in our equation have the same consistent uniform representation. Recall, that we started out with equations like (2.1), where all the quaternion parameters were given in one consistent representation—the elemental right-hand basis $\{1, i, j, k\}$. And we know that we may substitute any other right-hand basis representation, like $\{\boldsymbol{E}, \boldsymbol{I}_R, \boldsymbol{J}_R, \boldsymbol{K}_R\}$, for the basis elements in the equation's parameters, and obtain an equivalent statement of the problem, whose solution has the same component values.

Therefore, if we can write down and solve this equation entirely in the 4×4 matrix format, we can adopt the methods discovered, to solve the same problem once again, in the elemental, $\{1, i, j, k\}$, basis format, and so dispense with the matrix method altogether. The algebra of matrices then being considered simply the tool to discover just what is lacking in the quaternion algebra, that prevents us from solving these linear equations entirely within the quaternion system alone.

So, here's the essential idea. First we solve, $\boldsymbol{A}q = C$, $q = \boldsymbol{A}^{-1}C$, with q and C in column vector format. Then write, $q = q_0\boldsymbol{E} + q_1\boldsymbol{I}_R + q_2\boldsymbol{J}_R + q_3\boldsymbol{K}_R$, and see if we can find that new 4×4 parameter, $\boldsymbol{H}$, such that, $\boldsymbol{H}q = C$, where, $C = c_0\boldsymbol{E} + c_1\boldsymbol{I}_R + c_2\boldsymbol{J}_R + c_3\boldsymbol{K}_R$. Then work out a transformation between $\boldsymbol{H}$ and $\boldsymbol{A}$. That should help us discover the steps required to modify quaternion algebra to represent and solve this problem. We should be able to write, $\boldsymbol{H} = Cq^{-1}$, since q and C are now both 4×4 matrices. The q-coefficients are expressed in terms of c_k, h_j, while the $\boldsymbol{A}$-components are only in terms of h_j.

`Bilateral factor's Inverse formula`;

$$w_0 = +h_0^3 + h_0(+h_{R1}^2 + h_{R2}^2 + h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2)$$

$$w_{R1} = -h_{R1}^3 + h_{R1}(-h_0^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2)$$

$$w_{R2} = -h_{R2}^3 + h_{R2}(-h_{R1}^2 - h_0^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2)$$

$$w_{R3} = -h_{R3}^3 + h_{R3}(-h_{R1}^2 - h_{R2}^2 - h_0^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2)$$

$$w_{L1} = -h_{L1}^3 + h_{L1}(+h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_0^2 - h_{L2}^2 - h_{L3}^2)$$

$$w_{L2} = -h_{L2}^3 + h_{L2}(+h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_0^2 - h_{L3}^2)$$

$$w_{L3} = -h_{L3}^3 + h_{L3}(+h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 - h_0^2)$$

$w_{M1} = +2(+h_0h_{R1}h_{L1})$	$\boldsymbol{A}^{-1} =$
$w_{M2} = +2(+h_0h_{R2}h_{L2})$	$(w_0\boldsymbol{E}$
$w_{M3} = +2(+h_0h_{R3}h_{L3})$	$+w_{R1}\boldsymbol{I}_R + w_{R2}\boldsymbol{J}_R + w_{R3}\boldsymbol{K}_R$
$w_{A1} = +2(+h_0h_{R2}h_{L3})$	$+w_{L1}\boldsymbol{I}_L + w_{L2}\boldsymbol{J}_L + w_{L3}\boldsymbol{K}_L$
$w_{A2} = +2(+h_0h_{R3}h_{L1})$	$+w_{M1}\boldsymbol{I}_M + w_{M2}\boldsymbol{J}_M + w_{M3}\boldsymbol{K}_M$
$w_{A3} = +2(+h_0h_{R1}h_{L2})$	$+w_{A1}\boldsymbol{I}_A + w_{A2}\boldsymbol{J}_A + w_{A3}\boldsymbol{K}_A$
$w_{Z1} = +2(+h_0h_{R3}h_{L2})$	$+w_{Z1}\boldsymbol{I}_Z + w_{Z2}\boldsymbol{J}_Z + w_{Z3}\boldsymbol{K}_Z)/d$
$w_{Z2} = +2(+h_0h_{R1}h_{L3})$	where,
$w_{Z3} = +2(+h_0h_{R2}h_{L1})$	$d = h_0w_0 - \sum_k(h_{Rk}w_{Rk} + h_{Lk}w_{Lk})$

For example,

$$Cq^{-1} = \frac{(c_0E + c_1I_R + c_2J_R + c_3K_R)}{1}\frac{(q_0E - q_1I_R - q_2J_R - q_3K_R)}{q_0^2 + q_1^2 + q_2^2 + q_3^2} \tag{2.172}$$

So, either the c_k cancel each other in the formation of $\boldsymbol{H} = Cq^{-1}$, or we anticipate we'd have some kind of transformation from $\boldsymbol{A}$ to $\boldsymbol{H}$ that involves a C-type factor, like $\boldsymbol{H} = T\boldsymbol{A}$, or $\boldsymbol{H} = \boldsymbol{A}T$, or $\boldsymbol{H} = T\boldsymbol{A}T^{-1}$, or some combination of these, where T is a function of C only, $T = T(C)$. This investigative approach leads to some interesting insights, which the reader may explore. However, it helps to start with an even simpler problem, so here we shall follow a variation of this general method, which ends up transforming $\boldsymbol{A}q = C$ into $\boldsymbol{A} \otimes q = C$, with the introduction of a new multiplication operator $\otimes$, instead of $\boldsymbol{H}q = C$, or settling for the idea of keeping this dual representation for the quaternion parameters.

Consider a simplified bilateral factor equation;

$$(h_0\boldsymbol{E} + h_{R1}\boldsymbol{I}_R + h_{L1}\boldsymbol{I}_L)q = C \tag{2.173}$$

This equation has the form $\boldsymbol{A}q = C$, with the two quaternions, q and C, in column vector format, and the bilateral factor $\boldsymbol{A}$ is now the 4×4 matrix;

$$\boldsymbol{A} = h_0\boldsymbol{E} + h_{R1}\boldsymbol{I}_R + h_{L1}\boldsymbol{I}_L \tag{2.174}$$

To find the inverse of this hexpe number, we first define a new number that flips the signs on the imaginary units, very much like the idea of the conjugate,

$$g = h_0\boldsymbol{E} - h_{R1}\boldsymbol{I}_R - h_{L1}\boldsymbol{I}_L \tag{2.175}$$

then construct the product, $g\boldsymbol{A}$, to get,

$$g\boldsymbol{A} = (h_0^2 + h_{R1}^2 + h_{L1}^2)\boldsymbol{E} - 2h_{R1}h_{L1}\boldsymbol{I}_M \tag{2.176}$$

Recognizing the form $(a\boldsymbol{E} - b\boldsymbol{I}_M)$, we use the standard algebraic formula $(x^2 - y^2) = (x + y)(x - y)$ to complete the construction of the inverse, by first defining a complementary factor, $f = (a\boldsymbol{E} + b\boldsymbol{I}_M)$, so

$$f = (h_0^2 + h_{R1}^2 + h_{L1}^2)\boldsymbol{E} + 2h_{R1}h_{L1}\boldsymbol{I}_M \tag{2.177}$$

Then the product, $fg\boldsymbol{A}$, is proportional to $\boldsymbol{E}$,

$$\begin{aligned} fg\boldsymbol{A} &= (a^2 - b^2)\boldsymbol{E} && (2.178)\\ \texttt{where} & \\ a &= (h_0^2 + h_{R1}^2 + h_{L1}^2) && (2.179)\\ b &= 2h_{R1}h_{L1} && (2.180) \end{aligned}$$

Our inverse is then given by the product, fg, divided by the normalizing factor, $(a^2 - b^2)$, so that,

$$\boldsymbol{A}^{-1} = \frac{w_0\boldsymbol{E} + w_1\boldsymbol{I}_R + w_2\boldsymbol{I}_L + w_3\boldsymbol{I}_M}{a^2 - b^2} \tag{2.181}$$

where

$$\begin{aligned} a^2 - b^2 &= (h_0^2 + h_{R1}^2 + h_{L1}^2)^2 - 4h_{R1}^2h_{L1}^2 && (2.182)\\ \texttt{and,} & \\ w_0 &= h_0(h_0^2 + h_{R1}^2 + h_{L1}^2)\\ w_1 &= h_{R1}(h_{L1}^2 - h_{R1}^2 - h_0^2)\\ w_2 &= h_{L1}(h_{R1}^2 - h_{L1}^2 - h_0^2)\\ w_3 &= 2h_0h_{R1}h_{L1} \end{aligned}$$

If we define new coefficients $w'_k = w_k/(a^2 - b^2)$ where $k = 0, 1, 2$, we can write this inverse,

$$\boldsymbol{A}^{-1} = w'_0\boldsymbol{E} + w'_1\boldsymbol{I}_R + w'_2\boldsymbol{I}_L + w'_3\boldsymbol{I}_M \tag{2.183}$$

Now the solution to the equation becomes,

$$q = \boldsymbol{A}^{-1}C \tag{2.184}$$

$$q = (w'_0\boldsymbol{E} + w'_1\boldsymbol{I}_R + w'_2\boldsymbol{I}_L + w'_3\boldsymbol{I}_M)C \tag{2.185}$$

$$q = w'_0\boldsymbol{E}C + w'_1\boldsymbol{I}_RC + w'_2\boldsymbol{I}_LC + w'_3\boldsymbol{I}_MC \tag{2.186}$$

Since, q and C, are in column vector format, this is,

$$q = w'_0\boldsymbol{E}\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix} + w'_1\boldsymbol{I}_R\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix} + w'_2\boldsymbol{I}_L\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix} + w'_3\boldsymbol{I}_M\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix} \tag{2.187}$$

contracting the matrix products, $\boldsymbol{E}C$, $\boldsymbol{I}_RC$, $\boldsymbol{I}_LC$, and $\boldsymbol{I}_MC$, we have,

$$q = w'_0\begin{pmatrix}c_0\\c_1\\c_2\\c_3\end{pmatrix} + w'_1\begin{pmatrix}-c_1\\c_0\\-c_3\\c_2\end{pmatrix} + w'_2\begin{pmatrix}-c_1\\c_0\\c_3\\-c_2\end{pmatrix} + w'_3\begin{pmatrix}-c_0\\-c_1\\c_2\\c_3\end{pmatrix} \tag{2.188}$$

We can write this as the single column vector,

$$q = \begin{pmatrix} w'_0c_0 - w'_1c_1 - w'_2c_1 - w'_3c_0\\ w'_0c_1 + w'_1c_0 + w'_2c_0 - w'_3c_1\\ w'_0c_2 - w'_1c_3 + w'_2c_3 + w'_3c_2\\ w'_0c_3 + w'_1c_2 - w'_2c_2 + w'_3c_3 \end{pmatrix} \tag{2.189}$$

and this column vector is equivalent to the quaternion with right-hand elements $\{1, i, j, k\}$, so we have,

$$\begin{aligned} q &= (w'_0c_0 - w'_1c_1 - w'_2c_1 - w'_3c_0).1\\ &+ (w'_0c_1 + w'_1c_0 + w'_2c_0 - w'_3c_1).i\\ &+ (w'_0c_2 - w'_1c_3 + w'_2c_3 + w'_3c_2).j\\ &+ (w'_0c_3 + w'_1c_2 - w'_2c_2 + w'_3c_3).k \end{aligned} \tag{2.190}$$

we can re-arrange this equation,

$$\begin{aligned} q &= w'_0(c_0.1 + c_1.i + c_2.j + c_3.k)\\ &+ w'_1(-c_1.1 + c_0.i - c_3.j + c_2.k)\\ &+ w'_2(-c_1.1 + c_0.i + c_3.j - c_2.k)\\ &+ w'_3(-c_0.1 - c_1.i + c_2.j + c_3.k) \end{aligned} \tag{2.191}$$

which we recognize we can write,

$$\begin{aligned} q = {} & w'_0(c_0.1 + c_1.i + c_2.j + c_3.k)\\ &+ w'_1.i.(c_0.1 + c_1.i + c_2.j + c_3.k)\\ &+ w'_2(c_0.1 + c_1.i + c_2.j + c_3.k).i\\ &+ w'_3.i.(c_0.1 + c_1.i + c_2.j + c_3.k).i \end{aligned} \tag{2.192}$$

or,

$$q = w'_0C + w'_1.i.C + w'_2C.i + w'_3i.C.i \tag{2.193}$$

where now, q and C, are elemental quaternions. Since we can replace the elemental right-hand basis elements $\{1, i, j, k\}$ by any other representation, we can now rewrite this equation in terms of the corresponding matrix elements $\{\boldsymbol{E}, \boldsymbol{I}_R, \boldsymbol{J}_R, \boldsymbol{K}_R\}$, which gives,

$$\begin{aligned} & q = w'_0\boldsymbol{E}C + w'_1\boldsymbol{I}_RC + w'_2C\boldsymbol{I}_R + w'_3\boldsymbol{I}_RC\boldsymbol{I}_R\\ \texttt{with,} & \\ & q = q_0\boldsymbol{E} + q_1\boldsymbol{I}_R + q_2\boldsymbol{J}_R + q_3\boldsymbol{K}_R\\ & C = c_0\boldsymbol{E} + c_1\boldsymbol{I}_R + c_2\boldsymbol{J}_R + c_3\boldsymbol{K}_R \end{aligned} \tag{2.194}$$

We now have the solution written entirely in the 4×4 matrix representation format, and have eliminated all reference to the column vector 4×1 matrix representation.

To get a better picture of this transformation, let's add clarifying subscripts to our parameters to indicate the current representation being used. So, $q = q_{4\times1}$ and $C = C_{4\times1}$, will idicate quaternions in column vector form. While, $q = q_{4\times4}$ and $C = C_{4\times4}$, will tell us that the quaternions are in square matrix form. Equations (2.186) and (2.194) can then be re-written ($\boldsymbol{I_M} \equiv \boldsymbol{I_R I_L}$),

$$q_{4\times1} = w_0'\boldsymbol{E}C_{4\times1} + w_1'\boldsymbol{I}_R C_{4\times1} + w_2'\boldsymbol{I}_L C_{4\times1} + w_3'\boldsymbol{I}_R\boldsymbol{I}_L C_{4\times1}$$
$$q_{4\times4} = w_0'\boldsymbol{E}C_{4\times4} + w_1'\boldsymbol{I}_R C_{4\times4} + w_2' C_{4\times4}\boldsymbol{I}_R + w_3'\boldsymbol{I}_R C_{4\times4}\boldsymbol{I}_R$$

What we notice, is that when we change the representation, from the mixed 4×1 column vector and 4×4 square matrix state, to the pure 4×4 square matrix form, some of our parameters and terms convert directly to their corresponding forms in the alternate state, while others involve more substantial changes. The last two terms on the R.H.S. of these equations incur the transformation,

$$\boldsymbol{I}_L C_{4\times1} \mapsto C_{4\times4}\boldsymbol{I}_R \tag{2.195}$$

In other words, the left hand basis element, $\boldsymbol{I}_L$, moves over to the other side of the C parameter and changes its character to become a right hand basis element, $\boldsymbol{I}_R$ instead. This is the reverse of what happens in the initial steps when the B factor, in qB, moves over to the other side to give us that equivalent term, $B'q$. There, all the right-hand elements in B were replaced by left-hand elements to form B', so that the factor could be moved to the other side. That was necessary so that we could aggregate the factors on one side of the unknown, q. Now that we've found the solution, it's necessary to reverse that transformation, so that we can express everything in the right-hand basis again. After all, the problem was initially framed in terms of right-hand quaternions. The q, we seek, is a right-hand quaternion. It would be nice if we could indeed work out the solution entirely in one representation, which we'd obviously choose to be the 4×4 matrix.

But,

$$\boldsymbol{I}_L C_{4\times4} \neq C_{4\times4}\boldsymbol{I}_R \tag{2.196}$$
$$\boldsymbol{I}_L C_{4\times4} = C_{4\times4}\boldsymbol{I}_L \tag{2.197}$$

so, we'd have a conflict within our algebra, since, $\boldsymbol{I}_L$ already has a valid product with R-H quaternions in 4×4 matrix format—it commutes with these numbers.

To keep the representation the same, i.e. everything in 4×4 format, we'd have to try something like `splitting the multiplication operator into two forms`, for example,

$$\boldsymbol{I}_L \otimes C_{4\times4} \equiv C_{4\times4}\boldsymbol{I}_R \tag{2.198}$$
$$\boldsymbol{I}_L \cdot C_{4\times4} \equiv C_{4\times4}\boldsymbol{I}_L \tag{2.199}$$

Either we must split this operator into two, or we must split the representation into two.

`Splitting the representation` is the technique effectively employed by the matrix method to resolve this conflict with the two different interpretations of the product with left hand numbers. The single consistent elemental representation, $\{1, i, j, k\}$, is replaced by two different representations, a 4×4 square and a 4×1 column, and a single multiplication operator then suffices. This problem is, of course, a consequence of the non-abelian nature of the multiplication operator, $Z\cdot Y \neq Y\cdot Z$, which, by its very definition, introduces this duality into the specifications of multiplication. How do we write, $Z'\cdot Y = Y\cdot Z$, when, if $Y = Z$, we'd get $Z'\cdot Z = Z\cdot Z$, and thus, given that our algebra is an associative one, we must have, $Z'\cdot Z\cdot Z^{-1} = Z$, or $Z' = Z$, which tells us immediately that[15], $Z\cdot Y = Y\cdot Z$, contradicting the non-abelian definition of the product?

A non-abelian algebra can only resolve this paradox by splitting the operator or splitting the representation.

If we chose the path of the split multiplication operator, we'd need to find the right constructions for the specifications of these two products, $\cdot$ and $\otimes$, e.g;

$$\boldsymbol{I}_L \otimes C_{4\times4} \equiv C_{4\times4}\boldsymbol{I}_R \tag{2.200}$$
$$\boldsymbol{I}_R \otimes C_{4\times4} \equiv \boldsymbol{I}_R C_{4\times4} \tag{2.201}$$
$$\boldsymbol{I}_L \cdot C_{4\times4} \equiv C_{4\times4}\boldsymbol{I}_L \tag{2.202}$$
$$\boldsymbol{I}_R \cdot C_{4\times4} \equiv \boldsymbol{I}_R C_{4\times4} \tag{2.203}$$

where the $C_{4\times4}$ in these rules now represents any right hand quaternion written in 4×4 square matrix format. With the above definitions, we could write the usual eqn,

$$\boldsymbol{A}q_{4\times1} = C_{4\times1} \tag{2.204}$$

in the alternative form,

$$\boldsymbol{A} \otimes q_{4\times4} = C_{4\times4} \tag{2.205}$$

where every parameter is in the same consistent 4×4 matrix representation, with, now, `two ways` to multiply.

`Splitting the operator` is the path taken by the Heaviside-Gibbs "`Vector Algebra`", which replaces the single quaternion product with `dot` and `cross` products. But, Heaviside and Gibbs went even further and altered the definition of the basis elements to get positive squares, destroying the quaternion nature of the algebra in the process, instead of enhancing it with their extensions. Their type of operator splitting wasn't an effort to extend the algebra to solve quaternion problems. Rather, they sought to remove from quaternion algebra the features they felt were hindrances to the art. The operator splitting is, however, a much more complex process, so we shall take the simpler solution already presented by matrix algebra and adapt it instead.

Carets. We introduce the alternative concept of `'pivot variables'`, which are the quaternions about which parameter movements are made.

In our problems, these are the unknown variable and the inhomogeneous parameter of the linear equation.

When we move a quaternion factor around a pivot variable we either mark that pivot with a caret, or remove the caret, depending on which way the movements are made, and what types of quaternions are moving about. The caret is effectively an id substitute for the column vector.

Lets consider a simple example. Say we're given the following problem to solve. We're to find the expression for q, given that i, q, c are all right hand quaterions from the system $\{1, i, j, k\}$. We proceed as follows;

EXAMPLE 1:

$$(1+i)q + qi = c \qquad \Leftarrow \texttt{problem} \qquad (2.206)$$

$$\begin{aligned}
&(1+i_R)q + qi_R = c\\
&(1+i_R)\hat{q} + i_L\hat{q} = \hat{c}\\
&(1+i_R+i_L)\hat{q} = \hat{c}\\
&(1-i_R-i_L)(1+i_R+i_L)\hat{q} = (1-i_R-i_L)\hat{c}\\
&(3-2i_M)\hat{q} = (1-i_R-i_L)\hat{c}\\
&(3+2i_M)(3-2i_M)\hat{q} = (3+2i_M)(1-i_R-i_L)\hat{c}\\
&(9-4)\hat{q} = (3-3i_R-3i_L+2i_M-2i_Mi_R-2i_Mi_L)\hat{c}\\
&5\hat{q} = (3-i_R-i_L+2i_M)\hat{c}\\
&5\hat{q} = (3-i_R-i_L+2i_Ri_L)\hat{c}\\
&5\hat{q} = 3\hat{c}-i_R\hat{c}-i_L\hat{c}+2i_Ri_L\hat{c}\\
&5q = 3c-i_Rc-ci_R+2i_Rci_R\\
&5q = 3c-ic-ci+2ici
\end{aligned}$$

$$q = (3c-ic-ci+2ici)/5 \qquad \Leftarrow \texttt{solution} \qquad (2.207)$$

We mark our factors RIGHT-HAND, $i \mapsto i_R$; then move all known factors to the left, adding carets on the pivots, $q \mapsto \hat{q}$, $c \mapsto \hat{c}$, and converting the moving factors from R-H to L-H; then simplify the expressions to get the unknown $\hat{q}$ by itself on one side; then we express everything on the R.H.S in terms of R-H and L-H quaternions; and finally, we move all L-H factors to the right of the inhomogeneous parameter, $\hat{c}$, and change them into R-H, removing the carets; then we drop the R-H subscript labels, $i_R \mapsto i$, because everything is now in the R-H system again. That's it.

Note that we could work out this problem without reference to i_M, and instead keep the product of left and right terms in binary pair form, like i_Ri_L, but we must at least use both the left hand and right hand elements to solve the problem. All other basis elements of the hexpe algebra are generated by the R-H and L-H elements. So, these are all we need, operationally, to work out the solutions. But, the middle hand numbers—M,A,Z—are often convenient shorthand.

Consider another example, this time, instead of the form, $Aq + qB = c$, we illustrate form, $q + AqB = c$, with factors on both sides of the same q,

EXAMPLE 2:

$$q + 2kq(1+i) = c \qquad \Leftarrow \texttt{problem} \qquad (2.208)$$

$$\begin{aligned}
&q + 2k_Rq(1+i_R) = c\\
&\hat{q} + 2k_R(1+i_L)\hat{q} = \hat{c}\\
&(1+2k_R+2k_Ri_L)\hat{q} = \hat{c}\\
&(1+2k_R+2j_A)\hat{q} = \hat{c}\\
&(1-2k_R+2j_A)(1+2k_R+2j_A)\hat{q} = (1-2k_R+2j_A)\hat{c}\\
&(9+4j_A)\hat{q} = (1-2k_R+2j_A)\hat{c}\\
&(9-4j_A)(9+4j_A)\hat{q} = (9-4j_A)(1-2k_R+2j_A)\hat{c}\\
&(81-16)\hat{q} = (9-18k_R+18j_A-4j_A-8i_L-8)\hat{c}\\
&65\hat{q} = (1-18k_R-8i_L+14k_Ri_L)\hat{c}\\
&65\hat{q} = \hat{c}-18k_R\hat{c}-8i_L\hat{c}+14k_Ri_L\hat{c}\\
&65q = c-18k_Rc-8ci_R+14k_Rci_R\\
&65q = c-18kc-8ci+14kci
\end{aligned}$$

$$q = (c-18kc-8ci+14kci)/65 \qquad \Leftarrow \texttt{solution} \; (2.209)$$

Now let's consider a more elaborate example, with a form like (2.1), e.g. $A_1qB_1+A_2qB_2+A_3qB_3+A_4qB_4 = c$.

EXAMPLE 3:

$$q + 2iqi + jqj - kqk = c \qquad \Leftarrow \texttt{problem} \quad (2.210)$$

$$\begin{aligned}
&q + 2i_Rqi_R + j_Rqj_R - k_Rqk_R = c\\
&\hat{q} + 2i_Ri_L\hat{q} + j_Rj_L\hat{q} - k_Rk_L\hat{q} = \hat{c}\\
&(1+2i_M+j_M-k_M)\hat{q} = \hat{c}\\
&\hat{q} = 1/5\cdot(-1+4i_M-j_M+k_M)\hat{c}\\
&\hat{q} = 1/5\cdot(-1+4i_Ri_L-j_Rj_L+k_Rk_L)\hat{c}\\
&\hat{q} = 1/5\cdot(-\hat{c}+4i_Ri_L\hat{c}-j_Rj_L\hat{c}+k_Rk_L\hat{c})\\
&q = 1/5\cdot(-c+4i_Rci_R-j_Rcj_R+k_Rck_L)
\end{aligned}$$

$$q = (-c+4ici-jcj+kck)/5 \qquad \Leftarrow \texttt{solution} \quad (2.211)$$

Here we could either use our sign changing algebraic tricks to find the inverse of the the M-H number, or simply look up the previously given formula (2.53-54), with $\{h_0 = 1, h_{M1} = 2, h_{M2} = 1, h_{M3} = -1\}$, to obtain our inverse.

These examples are all simple enough to illustrate within a few lines of text, yet intricate enough to cover the basic ideas and demonstrate the power of the essential technique.

The Solution. With the mechanics of this operational method, we can now solve the original problem presented in equation (2.1),

$$A_1 q B_1 + A_2 q B_2 + ... + A_n q B_n = C \qquad (2.1)$$

without making any direct reference to matrix algebra, using, instead, the extended two-hand quaternion algebra we've developed.

First we apply the associative law for R-H quaternions, to group the pair products we're going to transform,

$$A_1(qB_1) + A_2(qB_2) + ... + A_n(qB_n) = C \qquad (2.212)$$

Then we convert the B-parameters to LEFT-HAND quaternions, B', and move them over to the left of the variable q, remembering to mark the q variables and inhomogeneous parameter C with the caret,

$$A_1(B'_1\hat{q}) + A_2(B'_2\hat{q}) + ... + A_n(B'_n\hat{q}) = \hat{C} \qquad (2.213)$$

Now we apply the associative law to group the known parameters together into pair products,

$$(A_1 B'_1)\hat{q} + (A_2 B'_2)\hat{q} + ... + (A_n B'_n)\hat{q} = \hat{C} \qquad (2.214)$$

Then we apply the distributive law to aggregate the known parameters and factor out the unknown variable,

$$(A_1 B'_1 + A_2 B'_2 + ... + A_n B'_n)\hat{q} = \hat{C} \qquad (2.215)$$

We determine whether this `hexpe number` in parenthesis has an inverse, and if so, multiply both sides by the inverse factor,

$$\hat{q} = (A_1 B'_1 + A_2 B'_2 + ... + A_n B'_n)^{-1}\hat{C} \qquad (2.216)$$

The inverse, if it exists, will have the general form,

$$\begin{aligned}(A_1 B'_1 + A_2 B'_2 + ... + A_n B'_n)^{-1} &= \\ P_1 Q'_1 + P_2 Q'_2 + ... + P_m Q'_m&\end{aligned} \qquad (2.217)$$

where the P_k are R-H quaternions, and Q'_k are L-H quaternions. So, we may write,

$$\hat{q} = (P_1 Q'_1 + P_2 Q'_2 + ... + P_m Q'_m)\hat{C} \qquad (2.218)$$

then using the distributive law we remove the parenthesis,

$$\hat{q} = (P_1 Q'_1)\hat{C} + (P_2 Q'_2)\hat{C} + ... + (P_m Q'_m)\hat{C} \qquad (2.219)$$

now we apply the associative law to group the L-H quaternions together with the inhomogeneous parameter,

$$\hat{q} = P_1(Q'_1\hat{C}) + P_2(Q'_2\hat{C}) + ... + P_m(Q'_m\hat{C}) \qquad (2.220)$$

and finally, we move the L-H quaternions, Q'_k, over to the right side of the inhomogeneous parameter, where they metamorphize into R-H quaternions, Q_k, letting us remove the carets, and since R-H quaternions are associative we can remove the parenthesis too, and we have,

$$q = P_1 C Q_1 + P_2 C Q_2 + ... + P_m C Q_m \qquad (2.221)$$

and we're done!

`Now a remark.` We started out with a problem stated entirely in Hamilton's right hand quaternions, and ended up with the solution which is again expressed in the same right hand quaternions. However, the intermediate steps required us to deviate into the domain of a higher dimensional hypercomplex number—in this case the two-hand extended quaternions—in order to work out the solution. This is entirely analogous to the classical mathematicians discovery that to arrive at the solutions to certain cubic equations over the reals, it was necessary to walk the path of the complex number for part of the way on the journey towards the real valued solution!

Singular Solutions. When the inverse (2.217) doesn't exist, we are unable to find a solution with our new method. That doesn't mean that there's no solution, however. Solutions to these singular equations may still exist. For example, $iq + qi = c$, results in a singular bilateral factor in, $(i_R + i_L)\hat{q} = \hat{c}$, which is easily seen, since $(-i_R - i_L)(i_R + i_L) = (2 - 2i_M)$, and, $(2 + 2i_M)(2 - 2i_M) = (4 - 4) = 0$, so $(i_R + i_L)$ has no inverse. Yet, if $c = 2$, then $q = -i$, is a solution. Or, if $c = 0$, then $q = j$ or $q = k$, or indeed any $q = aj + bk,\ a, b \in \mathbb{R}$, are all solutions. The multi-solution special case, $c = 0$, means that when $c = 2$, the complete solution is really, $q = -i + aj + bk,\ \forall\ a, b \in \mathbb{R}$. To find these singular solutions, then, we have to express the quaternions in terms of their components, and solve the resulting set of four linear equations using the usual methods of real algebra. The most general solution to the equation, $iq + qi = c$, is then found to be, $q = c_1/2 - c_0 i/2 + aj + bk,\ \forall\ a, b \in \mathbb{R}$, `when` $c_2 = c_3 = 0$, and there being no solution for q `if` $c_2 \neq 0$ `or` $c_3 \neq 0$.

Left Hand q. If we start out in (2.1) with left-hand quaternions instead of right-hand quaternions, we end up with very similar results. The equation, $Aq + qB = c$, with all parameters in L-H would be written, $A_L q_L + q_L B_L = c_L$, then moving the B_L parameter to the left of the q_L causes it to change into a R-H quaternion, $q_L B_L \mapsto B_R \hat{q}_L$, and similarly, $B_R \hat{c}_L \mapsto c_L B_L$. When we re-work the matrix representations given in (2.22-25), this time resolving the binary products with the left-hand basis, $ij = -k$, we get, $Aq = (a_0\mathbf{1} + a_1\boldsymbol{I}' + a_2\boldsymbol{J}' + a_3\boldsymbol{K}')q$, $qB = (b_0\mathbf{1} + b_1\boldsymbol{I} + b_2\boldsymbol{J} + b_3\boldsymbol{K})q$, which is the reverse of the right hand basis results. Note that we obtain exactly the same set of basis matrices, just reverse applications. The left hand matrices, $\{\mathbf{1}, \boldsymbol{I}', \boldsymbol{J}', \boldsymbol{K}'\}$, consistently represent L-H quaternions in either formulation; likewise, the right hand basis matrices, $\{\mathbf{1}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$, always represents R-H quaternions whether the column vector is R-H or L-H.

3. GROUP STRUCTURE.

An Extended Complex Number. Given that the `hexpe number` can simply be viewed as a particular decomposition of the 4×4 square matrix into an equivalent 4^2-dimensional hypercomplex number, one could obviously look for similar decompositions for other $N \times N$ square matrices where N has some value other than 4. The number $N = 4$, after all, was just the result of our search for that extension to the quaternion which would allow us to include both RIGHT-HAND and LEFT-HAND bases in one unified system. But, the idea, of partitioning a square matrix into a particular set of simple linearly independent matrices of same order that contain some other algebra, isn't really dependent, in general, on the quaternion system. As an example, we could also decompose the 2×2 square matrix as follows;

$$\boldsymbol{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3.1}$$

$$\boldsymbol{I} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \boldsymbol{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{K} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

This gives rise to the 4-dimensional hypercomplex algebra with the anti-commuting product rules,

$$\begin{aligned} \boldsymbol{E}^2 &= \boldsymbol{E}, \\ \boldsymbol{I}^2 &= -\boldsymbol{J}^2 = \boldsymbol{K}^2 = \boldsymbol{E}, \quad \boldsymbol{IJ} = -\boldsymbol{JI} = \boldsymbol{K}, \\ \boldsymbol{KI} &= -\boldsymbol{IK} = -\boldsymbol{J}, \quad \boldsymbol{JK} = -\boldsymbol{KJ} = \boldsymbol{I} \end{aligned} \tag{3.2}$$

These look a bit like quaternion basis elements, given that we have these same anti-commuting pairs, e.g. $\boldsymbol{IJ} = -\boldsymbol{JI}$. But, while all three pairs do anti-commute, unlike Hamilton's numbers, they do not follow the `cyclical` permuting rule among the three IJK axes. Two pairs follow the `cyclical` rule, $\boldsymbol{IJ} = +\boldsymbol{K}$, and, $\boldsymbol{JK} = +\boldsymbol{I}$, while the third is `acyclical`, $\boldsymbol{IK} = +\boldsymbol{J}$. It is almost as if the right-hand rule were being mixed with the left-hand rule in the same system. We seem to have RIGHT-LEFT-RIGHT, as we cyclically permute the IJK axes, one pair product being out of step with the others.

In addition to this, these imaginary elements are not all the square-roots of -1. In fact, only one, $\boldsymbol{J}$, represents the square-root of -1. The other two, $\boldsymbol{I}$ and $\boldsymbol{K}$, are roots of $+1$, instead. This mixing of the roots of $+1$ and -1, within the same 4-dimensional hypercomplex system, reminds us of Davenport's hypercomplex algebra. Only that, Davenport mixes two imaginary roots of -1 with one imaginary root of $+1$, while here we have somewhat the reverse situation, with two imaginary roots of $+1$ and one imaginary root of -1. And, of course, Davenport's elements commute, while these here anti-commute.

Notice again, also, that this root mixing alternates in sync with the alternating hand mentioned above—we have the root of $+1$, then of -1, then of $+1$, in harmony with the RIGHT-LEFT-RIGHT pattern found when cyclically permuting the IJK elements.

We can now proceed to write down the general hypercomplex number, h, its equivalent square matrix, $[a_{ij}]$, compare coefficients h_k and components a_{ij}, and apply the procedures, as described before, to construct the multiplicative inverse, h^{-1}.

$$h = h_0\boldsymbol{E} + h_1\boldsymbol{I} + h_2\boldsymbol{J} + h_3\boldsymbol{K} \tag{3.3}$$

$$h = [a_{ij}] = \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} \tag{3.4}$$

$$\begin{aligned} a_{00} &= +h_0 + h_1 \\ a_{10} &= +h_2 - h_3 \\ a_{01} &= -h_2 - h_3 \\ a_{11} &= +h_0 - h_1 \end{aligned} \tag{3.5}$$

$$\begin{aligned} h_0 &= (+a_{00} + a_{11})/2 \\ h_1 &= (+a_{00} - a_{11})/2 \\ h_2 &= (+a_{10} - a_{01})/2 \\ h_3 &= (-a_{10} - a_{01})/2 \end{aligned} \tag{3.6}$$

$$\begin{aligned} det([a_{ij}]) &= (a_{00}a_{11} - a_{10}a_{01}) \\ &= (h_0^2 - h_1^2 + h_2^2 - h_3^2) \end{aligned} \tag{3.7}$$

$$[a_{ij}]^{-1} = \frac{\begin{pmatrix} a_{11} & -a_{01} \\ -a_{10} & a_{00} \end{pmatrix}}{(a_{00}a_{11} - a_{10}a_{01})} \tag{3.8}$$

so,

$$h^{-1} = \frac{h_0\boldsymbol{E} - h_1\boldsymbol{I} - h_2\boldsymbol{J} - h_3\boldsymbol{K}}{(h_0^2 - h_1^2 + h_2^2 - h_3^2)} \tag{3.9}$$

This number, h, contains ordinary complex numbers as a sub-algebra, when $\{h_1 = 0, h_3 = 0\}$, i.e. $h = h_0\boldsymbol{E} + h_2\boldsymbol{J}$. So, we can consider this 4-dimensional hypercomplex number to be an extension to the complex number. The other two 2-dimensional sub-algebras, based on the alternate numbers, $h = h_0\boldsymbol{E} + h_1\boldsymbol{I}$ and $h = h_0\boldsymbol{E} + h_3\boldsymbol{K}$, do not form complex numbers—they involve imaginary numbers that are the roots of $+1$, instead of -1, and obey different rules, like not having a multiplicative inverse even when $h \neq 0$; and $gh = 0$, when $g \neq 0$ & $h \neq 0$.

Because of the alternating character in the patterns of the defining rules for this 4-dimensional hypercomplex number, we can just as well call this particular extension, more appropriately, the `alternating complex number`.

Whether we can construct useful and interesting hypercomplex numbers from square matrices of other orders remains a topic for further research.

Groups of Order 8. We recall that a Group is a set of elements $G = \{e, g_1, g_2, ..., g_{n-1}\}$ with a binary operation $\cdot$ defined on those elements, which satisfies the four rules: (1) `closure`—where $a \cdot b$ is in G, if a and b are in G; (2) `identity`—there exists a special unique element e, called the identity, where $e \cdot a = a \cdot e = a$, for all a in G; (3) `associativity`—for any three elements a, b, c we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$; and (4) `inverse`—every element a in G has an inverse companion a^{-1} in G, for which $a^{-1} \cdot a = a \cdot a^{-1} = e$. If in addition to these rules we have; (5) `commutativity`—for every pair of elements, a, b in G, $a \cdot b = b \cdot a$, then we call the Group `Abelian`, otherwise it's a `Non-Abelian` Group.

Well, the eight positive and negative quaternion basis elements $\{1, i, j, k, -1, -i, -j, -k\}$ form a Non-Abelian Group under the operation of multiplication. And the most useful 4-dimensional hypercomplex algebras we can construct will have four degrees of freedom $\{E, I, J, K\}$ whose eight $\pm$ unit values form a group of some kind.

These groups of eight elements, otherwise known as the `Groups of Order 8`, can only be formed in `five` distinct flavors. All other groups are isomorphic to one of these five typical groups. The usual names for these representative groups are: Q, for the quaternions; D_4, for the 4th dihedral group, which is the group of the symmetry transformations on the square; $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the triple product of the cyclic group of order 2; then, $\mathbb{Z}_2 \times \mathbb{Z}_4$, the product of cyclic groups of orders 2 and 4; and finally, $\mathbb{Z}_8$, the cyclic group of order 8. The first four groups are represented directly among the elements of the `hexpe numbers`, while the fifth can be constructed using linear combinations of the basis elements.

(1): Q $\{E, I_R, J_R, K_R\}$.

(2): D_4 $\{E, I_A, I_R, K_M\}$.

(3): $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$; $\{E, I_M, J_M, K_M\}$.

(4): $\mathbb{Z}_2 \times \mathbb{Z}_4$; $\{E, I_R, I_L, I_M\}$.

(5): $\mathbb{Z}_8$; $\{E, I_R, (E + I_R)/\sqrt{2}, (E - I_R)/\sqrt{2}\}$.

This basically means that the `hexpe algebra` contains several different other 4-dimensional hypercomplex sub-algebras. We not only have Hamilton's Quaternions, in R and L flavors, and those commuting middle-hand numbers—M, A, Z—which we've discussed at length, but a careful examination of the elements reveal that those commutative hypercomplex numbers studied by Davenport, and those 4-dimensional hypercomplex numbers we just introduced and referred to as `alternating complex numbers`, are also part of the `hexpe` system.

THE CAYLEY TABLES OF
THE FIVE GROUPS OF ORDER EIGHT.

(1):

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	1	0	6	7	5	4
3	3	2	0	1	7	6	4	5
4	4	5	7	6	1	0	2	3
5	5	4	6	7	0	1	3	2
6	6	7	4	5	3	2	1	0
7	7	6	5	4	2	3	0	1

(2):

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	7	6	4	5
2	2	3	0	1	5	4	7	6
3	3	0	1	2	6	7	5	4
4	4	6	5	7	0	2	1	3
5	5	7	4	6	2	0	3	1
6	6	5	7	4	3	1	0	2
7	7	4	6	5	1	3	2	0

(3):

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

(4):

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	0	5	6	7	4
2	2	3	0	1	6	7	4	5
3	3	0	1	2	7	4	5	6
4	4	5	6	7	0	1	2	3
5	5	6	7	4	1	2	3	0
6	6	7	4	5	2	3	0	1
7	7	4	5	6	3	0	1	2

(5):

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

The Product Tables of the group elements, also called Cayley Tables, are shown here in terms of the corresponding isomorphic forms using the binary + operation, with identity element 0. The set of the first n non-negative integers under addition modulo n forms a group called, $\mathbb{Z}_n = \{r : r = a + b \mod n;\ \texttt{with}\ a, b \in \mathbb{N}_0\}$, *which is isomorphic to the cyclic group of order n,* C_n, *and thus* $\mathbb{Z}_n$ *is often used in place of* C_n *in the naming of a group structure.*

***Q*:** This is the group of Hamilton's Quaternions. Although the RIGHT-HAND and LEFT-HAND quaternions are clearly distinct by virtue of their `ijk` *handedness*, the sets of basis elements from each of these numbers still form isomorphic groups. So, the R-H and L-H are both considered representations of the same typical group, Q.

The concept of isomorphism does not distinguish between right and left hands. The characteristic `ijk` sign is not sufficient to differentiate the left from the right, in this context, because it is an overall characteristic of the entire algebra, and not a measure that distinguishes different parts of the same algebra. Isomorphism simply checks whether the product tables of two algebras can be put into one-to-one correspondence with each other. If they can, the algebras are considered equal up to isomorphism. If not, they are different algebras from the point of view of isomorphism.

From a geometric point of view, the R-H and L-H algebras are easily distinguished, but an isomorphic test cannot tell the difference, since one can rearrange the entries in the R-H product table to obtain the L-H product table by taking suitable mirror images, or simply re-labeling the group elements appropriately.

For RIGHT-HAND quaternions, ij = +k, we can see the equivalence to the Cayley Table–(1), given above, with the label assignments;

$$\begin{array}{cccccccc} 1 & -1 & i & -i & j & -j & k & -k \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \tag{3.10}$$

For LEFT-HAND quaternions, ij = -k, we can see the equivalence to the Cayley Table–(1), given above, with the label assignments;

$$\begin{array}{cccccccc} 1 & -1 & -i & i & -j & j & -k & k \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \tag{3.11}$$

So, simply by inverting the signs of the imaginary elements, $\{+i, +j, +k\} \longrightarrow \{-i, -j, -k\}$, we can demonstrate the equivalence of the product tables for the R-H and L-H quaternions. This amounts to an inversion through the origin of coordinates for the `3-space` part of the quaternion, and is one geometric transformation operation that would turn a right-hand into a left-hand.

D_4**:** The group of basis elements of the `alternating complex number` defined in (3.2) is isomorphic to D_4, the 4th dihedral group, which is the group of symmetries of the `2-space` square. It is interesting that this 4-d hypercomplex number is the only one that can be formed from the decomposition of the 2×2 real matrix, all other 4-d hypercomplex numbers discussed here arise out of the decomposition of that higher order 4×4 matrix.

Now, the square with vertices 1-2-3-4 can be rotated by 90, 180, 270, and 360 degrees, to keep the same shape placement in 2-d space. The four vertices move around, but the orientation of the square is unchanged. These four rotations form a group, {**R0, R90, R180, R270**}, which is isomorphic to $\mathbb{Z}_4$, the cyclic group of order 4. The rotation by 360°, is the same as rotation by 0°, or no rotation at all, and is thus the identity element, **R0**.

The square can also be reflected in either of its two diagonals, {**D13, D24**}, or reflected in either of the two lines that bisect its opposite sides, {**B12, B23**}—the line that bisects side 1-2 also bisects side 3-4, while the line that bisects side 2-3 also bisects side 1-4, we only need **B12** and **B23** to indicate these two reflections. So, there are 4 more transformations that leave the square in place. These 4 rotations and 4 reflections form the group of eight elements called D_4.

·	R0	R90	R180	R270	D13	D24	B12	B23
R0	R0	R90	R180	R270	D13	D24	B12	B23
R90	R90	R180	R270	R0	B23	B12	D13	D24
R180	R180	R270	R0	R90	D24	D13	B23	B12
R270	R270	R0	R90	R180	B12	B23	D24	D13
D13	D13	B12	D24	B23	R0	R180	R90	R270
D24	D24	B23	D13	B12	R180	R0	R270	R90
B12	B12	D24	B23	D13	R270	R90	R0	R180
B23	B23	D13	B12	D24	R90	R270	R180	R0

By defining the binary · operation in this case to mean '`followed by`', so that the product expression **R90·R180** now means a rotation of 90° `followed by` a rotation of 180°, we can show Cayley Table-(2) equivalence, with the label assignments[16];

$$\begin{array}{cccccccc} \textbf{R0} & \textbf{R90} & \textbf{R180} & \textbf{R270} & \textbf{D13} & \textbf{D24} & \textbf{B12} & \textbf{B23} \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \tag{3.12}$$

We can then show that the `alternating complex number` defined in (3.2) has this same group structure, by making the label assignments;

$$\begin{array}{cccccccc} E & J & -E & -J & I & -I & K & -K \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \tag{3.13}$$

Now, this hypercomplex number is also a sub-algebra of our `hexpe number`. If we select the basis elements $\{E, -I_A, -I_R, -K_M\}$ from the `hexpe system`, and simply re-label these $\{E, I, J, K\}$, we can demonstrate that these elements form a group isomorphic to the `alternating complex number`. So, the D_4 group is also present in the generators of the `hexpe algebra`.

The general dihedral group, D_n, represents the group of symmetry operations on the regular n-sided polygon (or, regular `n-gon`), when $n > 2$, and has a group order of 2n. There are always n rotations and n reflections in this group. The equilateral triangle, D_3, for example, with vertex labels 1-2-3, has three rotations, {**R0, R120, R240**}, and three reflections, {**B23, B13, B12**} through the lines bisecting the sides at right angles, which makes for $2 \cdot 3 = 6$ elements. The square, D_4, of order $2 \cdot 4 = 8$, the pentagon, D_5, of order $2 \cdot 5 = 10$, and other regular `n-gons`, although possessing different group structures, all share one thing in common—they all represent symmetry groups characterizing `2-space`.

But, our `hexpe number` system is multidimensional. We've got sixteen degrees of freedom. And these sub-algebras under discussion are 4-dimensional hypercomplex numbers describing corresponding 4-d sub-spaces of our overall structure. What does it mean to find a `2-space` structure here among our 4-d numbers?

We might expect that 2-space to arise out of the intersection of spaces of some kind of higher order. And indeed, our dihedral hypercomplex number—$\{E, I_A, I_R, K_M\}$—is constructed from the intersection of three 4-d spaces, one quaternion R-H space $\{E, I_R, J_R, K_R\}$, and two middle-hand, M-H and A-H spaces, $\{E, I_M, J_M, K_M\}$ and $\{E, I_A, J_A, K_A\}$.

Think of the geometry of intersecting spaces. Two lines intersect in a point, which is 0-space. Two planes intersect in a line, which is 1-space. Two 3-spaces intersect in a plane, which is 2-space. Two 4-spaces intersect in a 3-space. There is always a reduction of 1 dimension when `two` spaces intersect. What about when `three` spaces intersect? Three lines generally don't even intersect. Three planes intersect at a point, which is 0-space. Three 3-spaces intersect in a line, which is 1-space. And three 4-spaces intersect in a plane, which is 2-space. And that's our dihedral hypercomplex number. There's a reduction of 2 dimensions when `three` spaces intersect. So, it seems as if our dihedral hypercomplex number, although it has 4 degrees of freedom, yet only exhibits the symmetries of a space with 2 degrees of freedom, because it is itself constructed from the intersection of 3 spaces with the type of symmetries more characteristic of higher dimensional spaces.

Also, "rotations" are only possible in the anti-commuting quaternion space, while reflections, inversions, and scale changes, are all that's available in the commuting middle-hand spaces. We've got one axis of rotation, and two axes of scale changes, in our hypercomplex number, $\{E, I_R, I_A, K_M\}$. Having only one axis of rotation immediately suggests to us we've got a plane structure. Rotations are governed by the anti-commuting rule for multiplications, and so are only available in non-abelian spaces. Q and D_4 are the only non-abelian groups among the five groups of order eight, the remaining three groups discussed below are abelian. Q follows the CYCLIC anti-commuting relation, $ij = -ji = +k, jk = -kj = +j, ki = -ik = +i$, for the R-H, or the ACYCLIC for the L-H, while D_4 follows the ALTERNATING anti-commuting relation, $ij = -ji = +k, jk = -kj = -j, ki = -ik = +i$, that mixes these two patterns of anti-commutation. Thus, Q, has 3 axes of rotation in the same sense. While, D_4 has that tension with its RIGHT-LEFT-RIGHT pattern—one right cancels the left, and effectively leaves the system with a single right-hand axes of rotation. Looked at this way, we can see why this hypercomplex number exhibits the symmetries more characteristic of the plane.

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$: The triple product of the cyclic group of order 2 is the characteristic group of the basis elements of the middle-hand numbers. This is also the symmetry group of the general cuboid. We recall the middle-hand numbers—M-A-Z—have quite different inverse formulas from the quaternions. Those cuboid weight factors that appear in the formulas are quite a striking difference from the simple sign changes in taking the quaternion conjugate. We can understand why the simple sign change works for the quaternion. We've just discussed the fact that an inversion in `3-space` turns a right hand into a left hand. And since multiplying by a quaternion involves a rotation, following such a product by another product with the conjugate reverses this operation because it automatically involves a rotation of the same angle now in the opposite sense. That leaves us with a pure magnitude change with no rotation. When we normalize to get the inverse, we also reverse the magnitude change, leaving us with a net scale factor of 1.

But, things are not so simple with the middle-hand. The process of constructing the inverse now involves volumetric expressions from three different rectangular boxes—the cube, the square cuboid, and the general cuboid. These boxes have the symmetry groups;

`cube`	O_h	$= S_4 \times \mathbb{Z}_2$	48
`sq cuboid`	D_{4h}	$= D_4 \times \mathbb{Z}_2$	16
`cuboid`	D_{2h}	$= D_2 \times \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	8

The cube has the hexoctahedral symmetry group, O_h, with 48 elements, which is the product of the symmetric group, S_4, of permutations of 4 objects, with order $4! = 24$, and the cyclic group $\mathbb{Z}_2$, with order 2. The square faced cuboid has symmetry group, D_{4h}, with a third as many elements, and is the product of square's dihedral group, D_4, and cyclic group, $\mathbb{Z}_2$. Weighing in with the smallest number of elements is the group of symmetries of the cuboid with only 8 members.

We recall that the middle hand number M-H generates a `nonproportional scaling` transformation, which results in shape shifting the volume space, where cubes become cuboids, spheres become ellipsoids, and so on. So, the inverse hexpe number here basically has to reverse these volumetric shape changes, and hence the relatively strange cuboid form for the weight factors—an indication that shapes of volumes are being adjusted. Note that these volume changes are also restricted to cuboid type scale changes, the shape shifts are not arbitrary—we couldn't do `Khufu's Transform`, for example. It should come as no surprise then, that the numbers that generate these transformations are based on the group of symmetry elements from the cuboid.

For the MIDDLE-HAND numbers, $\{\boldsymbol{E}, \boldsymbol{I}_M, \boldsymbol{J}_M, \boldsymbol{K}_M\}$, we can see the equivalence to the Cayley Table-(3), given above, with the label assignments;

$$\begin{matrix} \boldsymbol{E} & -\boldsymbol{E} & \boldsymbol{I}_M & -\boldsymbol{I}_M & \boldsymbol{J}_M & -\boldsymbol{J}_M & \boldsymbol{K}_M & -\boldsymbol{K}_M \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \tag{3.14}$$

By replacing the M subscript with either A or Z, we show all three middle-hand numbers have the same table.

Now, if we start out with a simple line segment, drawn somewhere in space, say with the vertex labels 1-2, and consider the symmetries, we see that there are only two possibilities. Either we leave the line segment unchanged, e : 1-2 $\rightarrow$ 1-2, or we reflect through the line's midpoint that bisects the segment, a. This forms a group with two elements, $\{e, a\}$. The reflection swaps the vertices a: 1-2 $\rightarrow$ 2-1, so that if we perform two reflections in succession the vertices return to their original places, and this means, $a \cdot a = e$, and we can just as well write, $a \cdot a = a^2$, for convenient shorthand notation. This is a cyclic group isomorphic to $\mathbb{Z}_2$.

·	e
e	e

$\mathbb{Z}_1$

·	e	a
e	e	a
a	a	e

$\mathbb{Z}_2$

·	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\mathbb{Z}_3$

In fact, the first few groups of small orders, i.e. with order 1, 2, or 3, can only be formed in one distinct way, and they are all cyclic groups. The first group, $\mathbb{Z}_1 = \{e\}$, has only one element, e, the identity that maps every point of a geometric object back onto itself. It represents the symmetry group of objects that have no symmetry at all, so that the only transformation that leaves the shape in place is the one that doesn't do anything.

These first three groups have orders that are all prime numbers (with 1 being considered a prime here). Whenever the order is prime there is one and only one group that can be constructed with that order. So there is only one form for the product table of prime order groups. This follows from `Cauchy's Theorem` for groups, which states that *for every prime p that divides the order n of a finite group, there will exist group elements of order p, and cyclic subgroups of order p.*

Therefore, we first encounter multiple groups at the order 4. Since, $4 = 2 \times 2$, there must be elements, $g^2 = e$, and subgroup, $\mathbb{Z}_2$, and there are two distinct groups that can be formed. These 4th order groups are;

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	a	e
c	c	b	e	a

$\mathbb{Z}_4$

·	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

$D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$

In constructing group tables, it helps to know that in each row and each column every element is represented exactly once. There are never two copies of any element in a row or column. Each row (or column) contains a distinct permutation of the elements, different from every other row (or column). This characteristic determines, at a glance, whether the Cayley Table for a set of elements is actually that of a mathematical group.

To construct a group of order 4, therefore, we start with the identity element, e, and add an order 2 element, say a with $a^2 = e$, which is required by Cauchy's Theorem. This gives us the first four table entries in the top-left corner of the Cayley Table. Then we add the third element, b, and determine that ab can't be either a or e, because that would duplicate an element in the same row, and ab can't be b, because that would duplicate a column element, so this must be the fourth element c. This permutation rule then again tells us that, $ba = c$, $ac = b$ and $ca = b$. That leaves us to just determine the four entries in the right-bottom square of the table: bb, bc, cb, cc.

Now, there are only two ways to resolve bb. This must be either a or e. Both lead to valid resulting tables. Once we choose one, the rest of the table can be filled in immediately using the permutation rule again. If we pick, $bb = a$, we must then have, $bc = e$, $cb = e$, and $cc = a$. This gives us a group isomorphic to $\mathbb{Z}_4$. But, if we pick, $bb = e$, then we must have, $bc = a$, $cb = a$, and $cc = e$. This gives us a group called the Kleins Four-Group, V, which is isomorphic to the dihedral group, D_2, and again to the direct product of a pair of cyclic groups of order 2, i.e. $\mathbb{Z}_2 \times \mathbb{Z}_2$. This latter group is the group of symmetries of the rectangle, which has half as many symmetry transformations as the square.

DIRECT PRODUCTS. The cartesian product of two sets, $G = \{g_1, g_2, ..., g_n\}$ and $H = \{h_1, h_2, ..., h_m\}$, with orders n and m, is defined as the set of ordered pairs, $G \times H = \{(g_1, h_1), (g_1, h_2), ..., (g_n, h_m)\}$, which has order $n \times m$. If these two sets form groups under the binary operations, $\circ$ and $\otimes$, respectively, then the "`direct product`" of these two groups, also written $G \times H$, is a new group under the composite binary operator $\cdot$ defined by,

$$G \times H = \{(g, h) \colon g \in G, h \in H\}$$

$$(g_1, h_1) \cdot (g_2, h_2) = (g_1 \circ g_2, h_1 \otimes h_2) \tag{3.15}$$

$$(g_1, h_1), (g_2, h_2) \in G \times H$$

Using, $\mathbb{Z}_2$, the cyclic group of order 2, given above for the line segment, we can then construct, $\mathbb{Z}_2 \times \mathbb{Z}_2$,

·	(e,e)	(a,e)	(e,a)	(a,a)
(e,e)	(e,e)	(a,e)	(e,a)	(a,a)
(a,e)	(a,e)	(e,e)	(a,a)	(e,a)
(e,a)	(e,a)	(a,a)	(e,e)	(a,e)
(a,a)	(a,a)	(e,a)	(a,e)	(e,e)

$\mathbb{Z}_2 \times \mathbb{Z}_2$

The rectangle only has two reflections and two rotations that leave the shape in place. The rotations are 0° and 180°. The latter being a 2-fold operation, like reflections, that is to say, two consecutive transformations return the vertices to their starting positions. So all the rectangle's non-trivial symmetries are 2-fold.

Groups of order 5 and 7, being prime orders, each only have one table construction. Notice that the group of order 3, given above, has no identity elements on the main diagonal other than, $e^2 = e$. There are no elements with order 2, since the prime number 2 doesn't divide the order of the group. So, after we pick the first two elements, $\{e, a\}$, we can't set $a^2 = e$, the only option is to set it to the third element, $a^2 = b$. After this, the permutation rule determines the remaining entries. A similar situation holds for the order 5 and 7 groups.

·	e	a	b	c	d
e	e	a	b	c	d
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c

$\mathbb{Z}_5$

·	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	b	c	d	f	g	e
b	b	c	d	f	g	e	a
c	c	d	f	g	e	a	b
d	d	f	g	e	a	b	c
f	f	g	e	a	b	c	d
g	g	e	a	b	c	d	f

$\mathbb{Z}_7$

Suppose we'd like to construct the Cayley Table for the group of order 5. Now, 2 doesn't divide 5, so $a^2 \neq e$, and by closure and the permutation rule, a^2 must be one of the other elements, different from both e and a. Let, $a^2 = b$, then $ab = a^3$, but $a^3 \neq e$, since 3 doesn't divide 5, so ab must be different from e, a, and b, therefore lets call it c. Since this is a cyclic group, and all cyclic groups are abelian, the table must be symmetric about the main diagonal, and so $ba = ab = c$. What about bb? It can't be e, b, or c. Now $bb \neq a$, because $bb = a^4$, and if $a^4 = a$, which can be re-written, $a \cdot a^3 = a \cdot e$, then the cancellation laws would require, $a^3 = e$, which we just mentioned is impossible. So, bb must be a new element, and therefore must be the final element d. Similarly, ac can't be a, b, or c, because they are already in the same row. It must be either e or d. But, if $ac = e$, then $a(ab) = e$, that is, $(aa)b = b^2 = e$, which is impossible. So, the only valid entry here is $ac = d$, from which the abelian character also tells us $ca = ac = d$. Then we can just complete the row, $ad = e$, and column, $da = e$. Then, $bc = b(ba) = (b^2)a = a^5$. Since there must be an order 5 element in this group, let this be one such element, so $bc = a^5 = e$, then $cb = bc = e$, also. Then we complete the column and row, $db = bd = a$. Now, $cc = (ab)c = a(bc) = a(e) = a$, and from here we can fill in the rest, $dc = cd = b$, and $dd = c$.

The table for the cyclic group of order 7 can be similarly determined. Of course, an even easier way to construct this table would be to use the modulo arithmetic results from adding numeric elements $\{0, 1, 2, 3, 4, 5, 6\}$, whereafter these can simply be relabelled $\{e, a, b, c, d, f, g\}$. But, by using the Group Axioms, Cauchy's Theorem, the permutation rule, and other established rules and results of group theory, we can construct any group, not just the cyclic groups. The cancellation law we used here comes in two forms. The *left cancellation law* states that whenever, $a \cdot b = a \cdot c$, we can cancel the identical left element to obtain, $b = c$. And the *right cancellation law* states that whenever, $b \cdot a = c \cdot a$, we can cancel the identical right element to obtain, $b = c$. These two cancellation laws may be used as alternate axioms, together with closure and associativity, in the definition of a group.

Group. A set G and a binary operation $\cdot$ is called a group if and only if one of the two sets of axioms hold,

=either=

1. `Closure:` $\forall\ a, b \in G \Rightarrow\ a \cdot b \in G$.
2. `Associativity:` $\forall\ a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. `Identity:` $\exists\ e \in G,\ a \cdot e = e \cdot a = e, \forall\ a \in G$.
4. `Inverse:` $\forall\ a \in G,\ \exists\ a^{-1} \in G,\ a^{-1} \cdot a = a \cdot a^{-1} = e$.

=or=

1. `Closure:` $\forall\ a, b \in G \Rightarrow\ a \cdot b \in G$.
2. `Associativity:` $\forall\ a, b, c \in G, (a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. `Right Cancel:` $b \cdot a = c \cdot a \iff b = c,\ \forall\ a, b, c \in G$.
4. `Left Cancel:` $a \cdot b = a \cdot c \iff b = c,\ \forall\ a, b, c \in G$.

Given one set of axioms we can derive the other. However, it is the cancellation laws that tell us that the product ab can't be either a or b, unless one or both of these are in fact the identity element e, and therefore we can't have more than one copy of an element in any row or column. For suppose there were two different elements, $b \neq c$, for which the products, ab and ac, produced the same element, d, in a given row. Then because, $ab = d = ac$, this means, $ab = ac$, and the cancellation law requires, $b = c$, which contradicts our requirement that $b \neq c$. So, there can be no copies of elements in any row (column), therefore every element of the group must be present in every row (column), and hence every row (or column) must be a distinct permutation of all the elements of the group.

There are two groups of order 6, the cyclic group, $\mathbb{Z}_6$, and the 3rd dihedral group, D_3. Notice that an even easier way to construct the cyclic group is to simply copy the elements from one row to the row below with a simultaneous shift of all elements to the left by one column. The permutations are truly cyclic!

·	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	c	d	f	e
b	b	c	d	f	e	a
c	c	d	f	e	a	b
d	d	f	e	a	b	c
f	f	e	a	b	c	d

$\mathbb{Z}_6$

·	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	f	d	e	b	a
d	d	c	f	a	e	b
f	f	d	c	b	a	e

$D_3 = S_3$

This observation allows us to immediately write down the table for the cyclic group $\mathbb{Z}_6$.

Now, given that, $6 = 2 \times 3$, there must be an order 2 element, e.g. $c^2 = e$, and an order 3 element, e.g. $b^3 = e$, and the two cyclic groups, $\mathbb{Z}_2$ and $\mathbb{Z}_3$, must be subgroups. This allows us to show that there are only two possible groups of order 6, the cyclic group, $\mathbb{Z}_6$, and the group of symmetries of the equilateral triangle, D_3, with its three rotations and three reflections, which also happens to be the smallest *non-abelian* group. Proceeding in a similar manner we can demonstrate that there are only `five` groups of order eight, and that these groups are represented by the Cayley Tables (1)-(5) given above.

SYMMETRIC GROUP, S_n. Every row in the product table for a group of order n is a permutation of its n elements. A group consisting of *some* permutations of the elements, and an operation doing one "followed by" the other is called a `Permutation Group`, while the group of ALL the permutations on n elements is called the `Symmetric Group` of *degree* n, and is written S_n, it has $n!$ order. Every permutation group is a subgroup of some symmetric group—A subset that also satisfies the group axioms is called a subgroup—and Cayley's Theorem states that every symmetric group of degree n will contain all possible groups of order n as subgroups.

$D_3 \equiv S_3$. The dihedral group of the triangle, D_3, is also isomorphic to the symmetric group, S_3. We can take the three vertices 1-2-3 of the triangle, as the entities being permuted, and illustrate this.

The usual convention is to represent the permutation operation by ordered tuples of the moving vertices, so that $(1, 2, 3)$ represents the cyclical permutation of labels in which $1 \mapsto 2, 2 \mapsto 3, 3 \mapsto 1$, while, $(1, 2)$ represents the exchange permutation, $1 \mapsto 2, 2 \mapsto 1$. Note that, since the vertex 3 doesn't move in this latter permutation, it is left out of the ordered tuple. Only moving labels are shown. Since the labels are all single digits here, we may drop the comma and abbreviate these two permutation operations, (123) and (12), respectively. The identity element can be written (1), interpreted $1 \mapsto 1$, while the other labels also stay on their original vertices. We could also write the identity either (2) or (3). The six permutation operators are then,

(1)	: 1-2-3 $\mapsto$ 1-2-3,	(12)	: 1-2-3 $\mapsto$ 2-1-3,
(123)	: 1-2-3 $\mapsto$ 3-1-2,	(13)	: 1-2-3 $\mapsto$ 3-2-1,
(132)	: 1-2-3 $\mapsto$ 2-3-1,	(23)	: 1-2-3 $\mapsto$ 1-3-2.

Then, $(12)(123) = (13)$, that is, (12) followed by (123) produces the same result as (13). The operation (12) maps 1-2-3 to 2-1-3, and then (123) maps this by cyclically permuting the labels to the right giving, 3-2-1.

·	(1)	(123)	(132)	(12)	(23)	(13)
(1)	(1)	(123)	(132)	(12)	(23)	(13)
(123)	(123)	(132)	(1)	(23)	(13)	(12)
(132)	(132)	(1)	(123)	(13)	(12)	(23)
(12)	(12)	(13)	(23)	(1)	(132)	(123)
(23)	(23)	(12)	(13)	(123)	(1)	(132)
(13)	(13)	(23)	(12)	(132)	(123)	(1)

S_3

R0	**R120**	**R240**	**B12**	**B23**	**B13**
(1)	(123)	(132)	(12)	(23)	(13)
e	a	b	c	d	f

This table gives the products for the group S_3, which can be identified with the rotations and reflections that make up the dihedral group, D_3, by making the corresponding assignments shown.

Given that there is only one way to construct the groups of the small orders, the first few symmetric, dihedral, and cyclic groups, are isomorphic, $S_1 = \mathbb{Z}_1$, $S_2 = D_1 = \mathbb{Z}_2$, $D_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$, $S_3 = D_3$.

S_4. Now consider the four vertices 1-2-3-4 of the regular tetrahedron. We can permute these labels in 4! ways, and thus describe the collection of changes with the symmetric group S_4. A tetrahedron has a total of 4 vertices and 6 edges, among its four equilateral triangular faces. Four of the vertices of a cube can also be selected to form a regular tetrahedron. This tetrahedron uses six of the cube's face diagonals for its edges, one diagonal from each square face. The other four vertices and six face diagonals of the cube form another tetrahedron, complementary to the first, such that these two inscribed tetrahedra intersect within the cubic volume and one is the inverse image of the other. This means that the symmetries of the regular tetrahedron are exactly half that of the cube, since they are represented by that collection of the cube's own symmetries which map one tetrahedron into itself, leaving out that other half that map one tetrahedron into the other. Since the tetrahedron's symmetry group is S_4, and a 2-fold inversion operation is $\mathbb{Z}_2$, the cube's full symmetry group is just the combination of these, $S_4 \times \mathbb{Z}_2$.

The volume of the intersection of the two inscribed tetrahedra forms a regular octahedron. Thus the octahedron is transformed into itself whenever the cube is transformed into itself, and visa versa, so both the cube and octahedron have the same symmetry group, called the `hexoctahedral group`, $O_h = S_4 \times \mathbb{Z}_2$.

These symmetry transformations are called `isometries`. An `isometry` is a geometric transformation that preserves distances between points of an object, and consists of rotations, reflections, inversions, and translations.

THE TETRAHEDRON, S_4. The 24 isometries of the regular tetrahedron can be broken down into 12 pure rotations, 6 pure reflections, plus 6 transformations that combine a quarter-turn rotation with reflection (or inversion); this latter rotation being the equivalent of a 90° turn about the face-to-face axis of the cube in which the tetrahedron is inscribed—the mirror plane bisects the cube perpendicular to this axis. These are,

R0	: 1-2-3-4	$\mapsto$	1-2-3-4 =	(1)
R1-120	: 1-2-3-4	$\mapsto$	1-4-2-3 =	(234)
R1-240	: 1-2-3-4	$\mapsto$	1-3-4-2 =	(243)
R2-120	: 1-2-3-4	$\mapsto$	4-2-1-3 =	(134)
R2-240	: 1-2-3-4	$\mapsto$	3-2-4-1 =	(143)
R3-120	: 1-2-3-4	$\mapsto$	4-1-3-2 =	(124)
R3-240	: 1-2-3-4	$\mapsto$	2-4-3-1 =	(142)
R4-120	: 1-2-3-4	$\mapsto$	3-1-2-4 =	(123)
R4-240	: 1-2-3-4	$\mapsto$	2-3-1-4 =	(132)
Ra-180	: 1-2-3-4	$\mapsto$	4-3-2-1 =	(23)(14)
Rb-180	: 1-2-3-4	$\mapsto$	3-4-1-2 =	(13)(24)
Rc-180	: 1-2-3-4	$\mapsto$	2-1-4-3 =	(12)(34)
P34	: 1-2-3-4	$\mapsto$	2-1-3-4 =	(12)
P14	: 1-2-3-4	$\mapsto$	1-3-2-4 =	(23)
P24	: 1-2-3-4	$\mapsto$	3-2-1-4 =	(13)
P23	: 1-2-3-4	$\mapsto$	4-2-3-1 =	(14)
P13	: 1-2-3-4	$\mapsto$	1-4-3-2 =	(24)
P12	: 1-2-3-4	$\mapsto$	1-2-4-3 =	(34)
C1-90	: 1-2-3-4	$\mapsto$	3-4-2-1 =	(2314)
C1-270	: 1-2-3-4	$\mapsto$	4-3-1-2 =	(1324)
C2-90	: 1-2-3-4	$\mapsto$	4-1-2-3 =	(1234)
C2-270	: 1-2-3-4	$\mapsto$	2-3-4-1 =	(1432)
C3-90	: 1-2-3-4	$\mapsto$	3-1-4-2 =	(1243)
C3-270	: 1-2-3-4	$\mapsto$	2-4-1-3 =	(1342)

. `The 24 Isometries of the Tetrahedron`, S_4.

First is the identity, **R0**, a rotation of 0°. Consider the vertex labels 1-2-3-4. The line drawn through vertex #1, which intersects the opposite triangular face at right angles, has two rotations, 120° and 240°, which we write **R1-120** and **R1-240**. These are permutations, (234) and (243). Now, if we draw the axis from the mid-point of edge 1-2 to the mid-point of edge 3-4, a 180° rotation, **Rc-180**, will swap labels 1 and 2, and also swap labels 3 and 4, so the permutation is (12)(34).

For the pure reflections, each mirror plane contains one edge and bisects the opposite edge. The plane, drawn through the vertices 3 and 4, contains the edge 3-4, and if this plane also bisects edge 1-2, it is then a plane of mirror symmetry. We write, **P34**, and the permutation operation is (12). Lastly, we consider that the tetrahedron is inscribed in a cube. Rotate that cube, by 90° or 270°, about a face-to-face axis, then reflect in the plane bisecting the cube perpendicular to this axis. Let us write one such operation pair, **C1-90** and **C1-270**, these then have permutation (2314) and (1324).

CUBE, $O_h = S_4 \times \mathbb{Z}_2$. The eight vertices of the cube form two complementary tetrahedra. When we invert the cube, by mapping each vertex to the corresponding vertex across the cubic diagonal, these tetrahedra exhange places. This inversion operation, **I**, has 2-fold symmetry, $\mathbb{Z}_2$, so by doubling tetrahedron's symmetry group we obtain the corresponding group of symmetries of the cube.

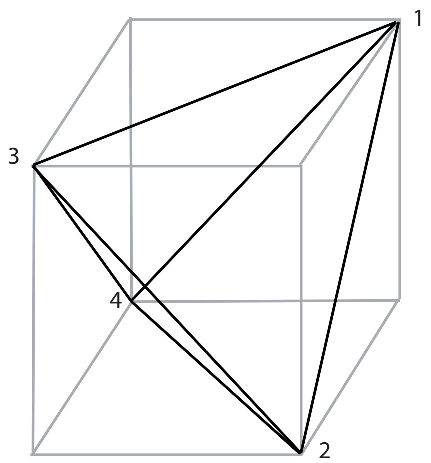

FIG. 3: THE TETRAHEDRON IN THE CUBE

SQUARE CUBOID, D_{4h}. When a plane figure with n-fold rotational symmetry is embedded in 3-space, the usual convention is to align that axis of rotation with the vertical. The plane is then horizontal, and a reflection in a line of the plane can now be interpreted either as a reflection in the vertical plane containing that line, or as a 180° rotation in 3-space with that line acting as the axis of rotation. When considering points restricted to the plane, these two operations produce the same result, but in 3-space these are very different operations that need to be distinguished; the first is written C_{nv} and the second, D_n. [17] In 3-space, all the $2n$ isometries of the dihedral group, D_n, are now rotations. To include the additional isometries that represent reflections, a subscript, h, v, d, is added to indicate whether the mirror plane is "horizontal", "vertical", or "diagonal", and these symmetry groups are then written, D_{nh}, D_{nv}, D_{nd}.

The dihedral group with horizontal mirror plane reflections can also be written, $D_{nh} = D_n \times \mathbb{Z}_2$, and has order $4n$. This is the symmetry group of the regular n-sided prism. When, $n = 4$, this is the symmetry group for the square cuboid, $D_{4h} = D_4 \times \mathbb{Z}_2$, which has order 16. When, $n = 2$, this is the symmetry group for the general cuboid, $D_{2h} = D_2 \times \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, with order 8.

Like the line segment, $\mathbb{Z}_2$, and the rectangle, $\mathbb{Z}_2 \times \mathbb{Z}_2$, the cuboid symmetry group is also built from 2-fold transformations, giving us the $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ group.

$\cdot$	(e,e,e)	(a,e,e)	(e,a,e)	(a,a,e)	(e,e,a)	(a,e,a)	(e,a,a)	(a,a,a)
(e,e,e)	(e,e,e)	(a,e,e)	(e,a,e)	(a,a,e)	(e,e,a)	(a,e,a)	(e,a,a)	(a,a,a)
(a,e,e)	(a,e,e)	(e,e,e)	(a,a,e)	(e,a,e)	(a,e,a)	(e,e,a)	(a,a,a)	(e,a,a)
(e,a,e)	(e,a,e)	(a,a,e)	(e,e,e)	(a,e,e)	(e,a,a)	(a,a,a)	(e,e,a)	(a,e,a)
(a,a,e)	(a,a,e)	(e,a,e)	(a,e,e)	(e,e,e)	(a,a,a)	(e,a,a)	(a,e,a)	(e,e,a)
(e,e,a)	(e,e,a)	(a,e,a)	(e,a,a)	(a,a,a)	(e,e,e)	(a,e,e)	(e,a,e)	(a,a,e)
(a,e,a)	(a,e,a)	(e,e,a)	(a,a,a)	(e,a,a)	(a,e,e)	(e,e,e)	(a,a,e)	(e,a,e)
(e,a,a)	(e,a,a)	(a,a,a)	(e,e,a)	(a,e,a)	(e,a,e)	(a,a,e)	(e,e,e)	(a,e,e)
(a,a,a)	(a,a,a)	(e,a,a)	(a,e,a)	(e,e,a)	(a,a,e)	(e,a,e)	(a,e,e)	(e,e,e)

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$

We can see the equivalence to Cayley Table-(3), given above, with the label assignments;

(e,e,e)	(a,e,e)	(e,a,e)	(a,a,e)	(e,e,a)	(a,e,a)	(e,a,a)	(a,a,a)
R0	**R1**	**R2**	**R3**	**P1**	**P2**	**P3**	**I**
0	1	2	3	4	5	6	7

GROUP PRODUCTS. The `direct product` concept can be extended to more than just two groups. For n groups, $G_1, G_2, \ldots, G_n$, each with binary operations, $o_1, o_2, \ldots, o_n$, the direct product is defined as the set of n-tuples, with binary operation $\circ$, given by,

$$G_1 \times G_2 \times \ \ldots \ \times G_n = \{(g_1, g_2, \ldots, g_n)\}$$
$$\text{where,} \quad g_1 \in G_1, g_2 \in G_2, \ldots g_n \in G_n.$$

and,

$$(g_1, g_2, \ldots, g_n) \circ (h_1, h_2, \ldots, h_n)$$
$$= (g_1 \circ_1 h_1, g_2 \circ_2 h_2, \ \ldots \ , g_n \circ_n h_n)$$

The product of these n groups is also a group, it has order, N, equal to the product of the orders of the groups in the product. The symbol, $|G|$, is used to denote the order of a group, G, so the group order can be written,

$$N = \left| \prod_{k=1}^{n} G_k \right| = |G_1 \times G_2 \times \ \ldots \ \times G_n| = \prod_{k=1}^{n} |G_k|.$$

If all the factor groups have the same order, $|G_k| = m$, then the order is just, $N = m^n$. If all the factors are the same group, $G_k = G$, we may write the product, G^n.

The cuboid, $\mathbb{Z}_2^3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, has four rotations, **R0, R1, R2, R3**, that leave the shape in place. One is the trivial identity rotation of 0° about any axis, and the others are all proper rotations of 180° about the three orthogonal axes of the space. Then, there are 3 reflections in the mirror planes, **P1, P2, P3**, perpendicular to these rotation axes. The last isometry is the inversion, **I**, that maps each vertex to the corresponding vertex across the cubic diagonal. The eight vertices, 1-2-3-4-5-6-7-8, are thus permuted into, 7-8-5-6-3-6-4-2, by the inversion operation, and we can write, **I** $\equiv$ (17)(28)(35)(46).

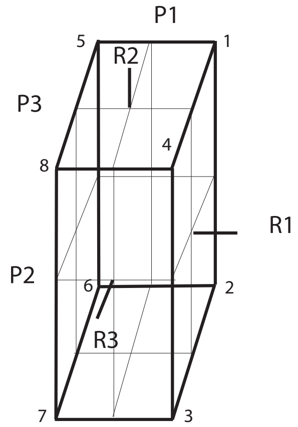

FIG. 4: SYMMETRIES OF THE CUBOID.

R0	1-2-3-4-5-6-7-8	(1)
R1	3-4-1-2-7-8-5-6	(13)(24)(57)(68)
R2	8-7-6-5-4-3-2-1	(18)(45)(27)(36)
R3	6-5-8-7-2-1-4-3	(16)(25)(38)(47)
P1	5-6-7-8-1-2-3-4	(15)(26)(37)(48)
P2	2-1-4-3-6-5-8-7	(12)(34)(56)(78)
P3	4-3-2-1-8-7-6-5	(14)(23)(58)(67)
I	7-8-5-6-3-6-4-2	(17)(28)(35)(46)

The permutation operations for the eight isometries are given in the above figure and table.

SPACE SYMMETRY. An interesting observation here is that the generators of the middle-hand numbers have the characteristic group, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, of the cuboid, which is a 3-space object. Yet, the M-A-Z numbers are 4-dimensional hypercomplex numbers. They have four degrees of freedom, and so describe a type of 4-space. But, it's a 4-space exhibiting 3-space type symmetries. This apparent reduction of one degree of freedom suggests that these numbers are in some way constructed from the intersection of higher dimensional spaces. In fact, recall that the middle-hand numbers are constructed from two quaternions. They are themselves generated from a pair of R-H and L-H quaternions. The quaternions are the genuine 4-d entities. When we combine two quaternions we obtain the middle-hand numbers, but with a reduction of one degree of freedom reflected in the symmetry group that characterize these new numbers. Then, when we combine a quaternion with two of these middle hand numbers, we again get another reduction of the degree of freedom reflected in the symmetry group that characterize the resulting number—those dihedral hypercomplex numbers have symmetry group, D_4, which are more characteristic of 2-space. This is the symmetry group of the square. So, while it is also possible to describe the `hexpe algebra` as being constructed from any two of the five R-L-M-A-Z numbers, it seems more natural to choose the R-L quaternion pair to be the generators, because of this *hierarchical structure in the space symmetry.*

$\mathbb{Z}_2 \times \mathbb{Z}_4$**:** Davenport's numbers. The commutative hypercomplex numbers studied by Davenport, and given here in equations (2.33), can be seen to form the group isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_4$ if we make the assignments,

$$\begin{matrix} E & I & -E & -I & K & -J & -K & J \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \tag{3.16}$$

and compare the result with the Cayley Table-(4) above.

Now, if we take the direct product of the two cyclic groups, $\{e, a\}$ and $\{e, b, b^2, b^3\}$, i.e. groups isomorphic to $\mathbb{Z}_2$ and $\mathbb{Z}_4$, we obtain,

$\cdot$	(e,e)	(e,b)	(e,b^2)	(e,b^3)	(a,e)	(a,b)	(a,b^2)	(a,b^3)
(e,e)	(e,e)	(e,b)	(e,b^2)	(e,b^3)	(a,e)	(a,b)	(a,b^2)	(a,b^3)
(e,b)	(e,b)	(e,b^2)	(e,b^3)	(e,e)	(a,b)	(a,b^2)	(a,b^3)	(a,e)
(e,b^2)	(e,b^2)	(e,b^3)	(e,e)	(e,b)	(a,b^2)	(a,b^3)	(a,e)	(a,b)
(e,b^3)	(e,b^3)	(e,e)	(e,b)	(e,b^2)	(a,b^3)	(a,e)	(a,b)	(a,b^2)
(a,e)	(a,e)	(a,b)	(a,b^2)	(a,b^3)	(e,e)	(e,b)	(e,b^2)	(e,b^3)
(a,b)	(a,b)	(a,b^2)	(a,b^3)	(a,e)	(e,b)	(e,b^2)	(e,b^3)	(e,e)
(a,b^2)	(a,b^2)	(a,b^3)	(a,e)	(a,b)	(e,b^2)	(e,b^3)	(e,e)	(e,b)
(a,b^3)	(a,b^3)	(a,e)	(a,b)	(a,b^2)	(e,b^3)	(e,e)	(e,b)	(e,b^2)

$\mathbb{Z}_2 \times \mathbb{Z}_4$

We can confirm the equivalence to Cayley Table-(4), given above, with the label assignments;

$$\begin{matrix} (e,e) & (e,b) & (e,b^2) & (e,b^3) & (a,e) & (a,b) & (a,b^2) & (a,b^3) \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}$$

These hypercomplex numbers are also contained in the `hexpe system`. If we take the basis elements $\{\boldsymbol{E}, \boldsymbol{I}_R, \boldsymbol{I}_L, \boldsymbol{I}_M\}$, and re-label these $\{E, I, J, K\}$, we'll obtain Davenport's numbers (2.33). So, the $\mathbb{Z}_2 \times \mathbb{Z}_4$ group is also contained in the `hexpe algebra`. The basis elements of general `hexpe number` can then be re-arranged, so that the number (2.34) can also be written,

$$\begin{aligned} h \;=\; & h_0\mathbf{E} \\ +\; & h_{R1}\mathbf{I_R} + h_{L1}\mathbf{I_L} + h_{M1}\mathbf{I_M} \\ +\; & h_{R2}\mathbf{J_R} + h_{L2}\mathbf{J_L} + h_{M2}\mathbf{J_M} \\ +\; & h_{R3}\mathbf{K_R} + h_{L3}\mathbf{K_L} + h_{M3}\mathbf{K_M} \\ +\; & h_{A1}\mathbf{I_A} + h_{A2}\mathbf{J_A} + h_{A3}\mathbf{K_A} \\ +\; & h_{Z1}\mathbf{I_Z} + h_{Z2}\mathbf{J_Z} + h_{Z3}\mathbf{K_Z} \end{aligned} \tag{3.17}$$

Each line below the $h_0\boldsymbol{E}$ term now contains a triplet from a commutative hypercomplex number. Therefore, a different view into the `hexpe number` reveals it to consist of 3 Davenport type commutative hypercomplex numbers, plus the 2 middle-hand commutative hypercomplex numbers, A-H and Z-H. Although written as five commutative numbers, these separate numbers do not actually commute with each other. Nevertheless, equation (3.17) shows one way that Davenport's algebra is contained within the `hexpe algebra`.

Now let any of the three triplets, $\{\boldsymbol{I}_R, \boldsymbol{I}_L, \boldsymbol{I}_M\}$, $\{\boldsymbol{J}_R, \boldsymbol{J}_L, \boldsymbol{J}_M\}$, $\{\boldsymbol{K}_R, \boldsymbol{K}_L, \boldsymbol{K}_M\}$, be re-labeled, $\{I, J, K\}$, then we can write the Davenport style `hexpe number`,

$$h = wE + xI + yJ + zK \tag{3.18}$$

$$\begin{aligned} &= w(EE) + x(IE) + y(-IK) + z(EK) \\ &= (wE + xI)E + (zE - yI)K \end{aligned}$$

This can be re-arranged into the form,

$$\begin{aligned} h = {} & [(wE + xI) - (zE - yI)](\frac{E-K}{2}) \\ & + [(wE + xI) + (zE - yI)](\frac{E+K}{2}) \end{aligned} \tag{3.19}$$

or, equivalently,

$$\begin{aligned} h = {} & [(w-z)E + (x+y)I](\frac{E-K}{2}) \\ & + [(w+z)E + (x-y)I](\frac{E+K}{2}) \end{aligned} \tag{3.20}$$

We can define two special unit numbers,

$$e_1 = \frac{E-K}{2}, \qquad e_2 = \frac{E+K}{2} \tag{3.21}$$

which then have the product rules,

$$e_1^2 = e_1, \qquad e_2^2 = e_2, \qquad e_1e_2 = e_2e_1 = 0 \tag{3.22}$$

Now let, $\{\xi, \eta\}$, be the two complex numbers,

$$\xi = [(w-z)E + (x+y)I] \tag{3.23}$$

$$\eta = [(w+z)E + (x-y)I] \tag{3.24}$$

then we can write the (3.18) hypercomplex number,

$$h = \xi e_1 + \eta e_2 \tag{3.25}$$

Davenport calls this the *canonical form* of the commutative hypercomplex number.

If we wanted to find the inverse of this number, we could start with the form in (3.18), flip a pair of signs to get our usual, g, number,

$$g = wE - xI - yJ + zK \tag{3.26}$$

whence, the product, gh, then yields,

$$gh = (w^2 + x^2 + y^2 + z^2)E + (2wz - 2xy)K \tag{3.27}$$

Recognizing this has the form, $(aE + bK)$, we define a new factor, f, with the complementary form, $(aE - bK)$, i.e.,

$$f = (w^2 + x^2 + y^2 + z^2)E - (2wz - 2xy)K \tag{3.28}$$

then, because, $(aE - bK)(aE + bK) = (a^2E - b^2E) = (a^2 - b^2)E$, we have,

$$fgh = (a^2 - b^2)E \tag{3.29}$$

`where`,

$$\begin{aligned} a &= (w^2 + x^2 + y^2 + z^2) \\ b &= (2wz - 2xy) \end{aligned} \tag{3.30}$$

Thus, our inverse, h^{-1}, is given by the product, fg, divided by this normalizing factor, $(a^2 - b^2)$, resulting in,

$$h^{-1} = \frac{w_0E + w_1I + w_2J + w_3K}{a^2 - b^2} \tag{3.31}$$

where,

$$\begin{aligned} a^2 - b^2 &= w^4 + x^4 + y^4 + z^4 + 2w^2(x^2 + y^2 - z^2) \\ &\quad - 2x^2y^2 + 2x^2z^2 + 2y^2z^2 + 8wxyz \end{aligned} \tag{3.32}$$

$$\begin{aligned} w_0 &= +w^3 + w(x^2 + y^2 + z^2) - z(2wz - 2xy) \\ w_1 &= -x^3 - x(w^2 + y^2 + z^2) - y(2wz - 2xy) \\ w_2 &= -y^3 - y(x^2 + w^2 + z^2) - x(2wz - 2xy) \\ w_3 &= +z^3 + z(x^2 + y^2 + w^2) - w(2wz - 2xy) \end{aligned}$$

But, we could start with Davenport's *canonical form* in (3.25), say the inverse, h^{-1}, is given by,

$$h^{-1} = \xi' e_1 + \eta' e_2 \tag{3.33}$$

then, because, $e_1e_2 = e_2e_1 = 0$ etc.., we have,

$$h^{-1}h = (\xi' e_1 + \eta' e_2)(\xi e_1 + \eta e_2) \tag{3.34}$$

$$= \xi'\xi e_1 + \eta'\eta e_2 \tag{3.35}$$

then, since, $e_1 + e_2 = E$, our inverse is obtained when, $\xi' = \xi^{-1}$ and $\eta' = \eta^{-1}$, so that,

$$h^{-1} = \xi^{-1}e_1 + \eta^{-1}e_2 \tag{3.36}$$

Using the complex conjugates we can write this,

$$h^{-1} = \frac{\xi^*\eta\eta^*e_1 + \eta^*\xi\xi^*e_2}{\xi\xi^*\eta\eta^*} \tag{3.37}$$

Then, substituting the definitions for these parameters, we get our normalizing factor,

$$\begin{aligned} (a^2 - b^2) &= (a - b)(a + b) \\ &= \xi\xi^*\eta\eta^* \\ &= [(w - z)^2 + (x + y)^2][(w + z)^2 + (x - y)^2] \end{aligned} \tag{3.38}$$

and the numerator resolves to the same results given by the w_k weight factors in (3.32) above.

$\mathbb{Z}_8$**:** The cyclic group of order 8, is the group of rotation symmetries of the regular octagon. All cyclic groups can be represented by the rotation symmetries of corresponding regular polygons, i.e. $\mathbb{Z}_n \subset D_n$. The `hexpe algebra` directly contains four of the five groups of order 8, only this group, $\mathbb{Z}_8$, is not represented directly among the basis elements. This is because our hypercomplex system is built from elements that are representations of the square-roots of $+1$ or -1. So, every basis element has order 1, 2, or 4 (with $\boldsymbol{E}^1 = \boldsymbol{E}, \boldsymbol{I}_M{}^2 = \boldsymbol{E}, ..., \boldsymbol{I}_R{}^4 = \boldsymbol{E}, ...$ etc..). Therefore, by design, we don't have an element with order 8. However, we may use linear combinations of basis elements to construct the group elements for $\mathbb{Z}_8$.

We only need to find a suitable 8-th order element. One such example gives the set of positives and negatives of the four matrices

$$\{\ \boldsymbol{E},\ \boldsymbol{I}_R,\ \frac{(\boldsymbol{E} + \boldsymbol{I}_R)}{\sqrt{2}},\ \frac{(\boldsymbol{E} - \boldsymbol{I}_R)}{\sqrt{2}}\ \}$$

A cyclic generator for this group is, $a = (\boldsymbol{E} + \boldsymbol{I}_R)/\sqrt{2}$, and the group elements are then the usual $\{E, a, a^2, a^3, a^4, a^5, a^6, a^7\}$. We can demonstrate the equivalence to Cayley Table-(5), given above, with the label assignments,

E	$\left(\frac{E+I_R}{\sqrt{2}}\right)$	I_R	$\left(\frac{-E+I_R}{\sqrt{2}}\right)$	$-E$	$\left(\frac{-E-I_R}{\sqrt{2}}\right)$	$-I_R$	$\left(\frac{E-I_R}{\sqrt{2}}\right)$
e	a	a^2	a^3	a^4	a^5	a^6	a^7
0	1	2	3	4	5	6	7

THE HEXPE GROUP. Apart from these groups of order eight, the entire collection of $\pm$ `hexpe` basis elements form a non-abelian group of order 32.

Group Statistics. Given that the `hexpe system` contains 15 imaginary basis elements, which form various triplets on which different types of 4-d hypercomplex numbers can be built, the question naturally arises, how many different ways can each type of number be constructed by selecting an appropriate triplet? We already know that these numbers can only be from one of the four groups: Q, D_4, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, and $\mathbb{Z}_2 \times \mathbb{Z}_4$. But, exactly how many times are these groups represented among the 15 elements? Now there are $\frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} = 455$ ways to pick 3 from 15. But, how many of these form a closed $\{I, J, K\}$ triple, i.e. where $IJ = \pm K$? From the `hexpe` product table (TABLE T.2) we can see that there are $(15 \cdot 15 - 15)/2 = 105$ distinct pairs that produce the required third element for such a closed triple. The defining product rules for each triple use up three of these pairs, e.g. $IJ = \pm K, JK = \pm I, KI = \pm J$. These $105/3 = 35$ unique triples are broken down into: 2 of Q, the R and L quaternions; 9 Davenport triples $\mathbb{Z}_2 \times \mathbb{Z}_4$, which require two 4th order elements each, so need to be constructed from one R and one L element, which can only be done in 3×3 ways; $\frac{1}{3}(\frac{9 \cdot 6}{1} \cdot \frac{2}{3} + \frac{9 \cdot 9 - 9}{2} \cdot \frac{1}{2}) = 18$ D_4 triples, which all have exactly one 4th order element taken from either R or L; and the remaining $\frac{9 \cdot 9 - 9}{2 \cdot 2 \cdot 3} = 6$ triples are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, which are the various ways of selecting the right combinations of the 9 M-A-Z elements.

4. APPLICATIONS.

The Sixteen Degrees of Freedom. One good thing about Hamilton's quaternions is that they have just the four degrees of freedom required to represent points in spacetime. But, what are we to make of the 16 degrees of freedom in the `hexpe numbers`? Well, these are degrees of transformation, not points in spacetime. To express the transformation of a 4-d spacetime event into another 4-d spacetime event, we need 16 degrees of freedom. That's the job of the 4×4 matrix. It tells us how these things change. It is the algebra of the transformation operations therefore that is described by the new algebra of the HEXPENTAQUATERNIONS.

The agents of transformation require 16 parameters to fully express all possible changes between two 4-d variables. Each of the four dependent variables, $v_k;\ k = 0, 1, 2, 3$, have those four initial independent variables for influence, $v_k(u_0, u_1, u_2, u_3);\ k = 0, 1, 2, 3$, leading to 16 parameters of change, $\partial v_k/\partial u_j;\ j, k = 0, 1, 2, 3$. We could just use the familiar 4×4 matrix algebra to describe the changes. But, what the `hexpe system` does is break down the transformation into special types of operations revealing the sub-structure of the components-of-change based on symmetry operations. Each of the five groups of order eight represents a unique symmetry type that is encapsulated in the sub-algebra built around that group.

We've seen how the MIDDLE-HAND numbers describe `shape shifting`, just like how RIGHT-HAND and LEFT-HAND quaternions describe `rotations`. So, when we decompose a 4×4 transformation matrix into an hexpe number, we get a sense of how much shape shifting verses how much rotation is involved in the operation.

Now, 4×4 matrix transformation operations are applicable in many areas of geometry and physics. The hexpe number is therefore versatile. Although we constructed the hexpe number using Hamilton's quaternions for a starting point, it really doesn't matter what the four-parameter variable undergoing transformation is. It doesn't have to be a quaternion at all. This is because the set of 16 basis elements that comprise the hexpe number form a complete set of linearly independent matrices of the same 4×4 order as the general transformation matrix being decomposed, and so any type of four-parameter variable, from any theory that uses such things, can benefit from the insights achieved by re-representation using this new hypercomplex algebra.

Affine Transformations. One of the more general types of useful transformations often employed in the study of 3-space comes from `Affine Geometry`.

$$F\colon \boldsymbol{x} \longrightarrow \boldsymbol{y} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{c} \tag{4.1}$$

An `affine transformation`, F, is a general linear transformation, $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$, plus a translation, $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{c}$.

This transformation preserves parallelism, which is to say, parallel lines are transformed into parallel lines, although lengths and angles within shapes may change. The cuboid scale changes generated by the middle-hand `hexpe numbers` are an example of such transformations. In the 3-space, the affine transformation can be represented by 3×3 matrix multiplication, followed by the addition of a 3×1 column vector.

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \tag{4.2}$$

This type of "inhomogeneous" equation, however, is often converted into a more convenient "homogenous" equation, by using a simple operational trick. By adding a fourth coordinate parameter, and re-writing the equation using 4-dimensional matrices, we can homogenize the equation, so that all affine transformations can be represented just by matrix multiplication alone.

HOMOGENEOUS COORDINATES.

$$\begin{pmatrix} 1 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ c_1 & a_{11} & a_{12} & a_{13} \\ c_2 & a_{21} & a_{22} & a_{23} \\ c_3 & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{4.3}$$

This is accomplished by adding a 1 to the column vectors, and then including the inhomogeneous term as the extra column of the expanded square matrix, filling in the rest of the extra matrix entries with 0s. The fourth coordinate parameter is therefore a fixed number, the value 1 being the most appropriate choice, although other fixed values may be chosen depending on the manner in which overall scaling operations are being included.

The 4th parameter may appear anywhere in the column, but the convention is to place this extra 1 either at the bottom or top of the column vector, and to modify the square matrix in corresponding fashion. The expanded coordinates are then referred to as the "`homogeneous coordinates`." In this way, the "inhomogeneous" equation in 3-dimensions, is converted into an "homogenous" equation in 4-dimensions. The geometry of 3-space being then described by special transformations in 4-space. This is somewhat reminiscent of the situation in Hamilton's quaternions, where the extra 4th parameter—the scalar—is included, more to facilitate algebraic calculations, rather than express the interaction of some real 4-dimensional coordinate of the space. Nevertheless, both Hamilton's 4th parameter, and the Affine Geometry's homogeneous 4th coordinate, are found to have appropriate meaningful interpretations in certain applications, where they are not just treated as mere operational techniques.

Hamilton's scalar takes on interpretive meaning when quaternions are applied to describe spacetime, and Affine Geometry's homogenous 4th coordinate takes on interpretive meaning in the field of Projective Geometry.

The key point of interest for the moment, however, is that affine geometry makes convenient use of 4×4 matrices to describe operations in 3-space geometry. The `hexpe algebra` presents an alternative way to view that 4×4 matrix algebra. So, we can use our new hypercomplex numbers to describe affine operations in the geometry of 3-space. Starting with the transformation matrix in (4.3), adapting the equations from (2.37), gives us,

$$
\begin{aligned}
h_0 &= (+1 + a_{11} + a_{22} + a_{33})/4 \\
\\
h_{M1} &= (-1 - a_{11} + a_{22} + a_{33})/4 \\
h_{M2} &= (-1 + a_{11} - a_{22} + a_{33})/4 \\
h_{M3} &= (-1 + a_{11} + a_{22} - a_{33})/4 \\
\\
h_{A1} &= (+c_1 + 0 - a_{32} - a_{23})/4 \\
h_{A2} &= (+c_2 - a_{31} + 0 - a_{13})/4 \\
h_{A3} &= (+c_3 - a_{21} - a_{12} + 0)/4 \\
\\
h_{Z1} &= (-c_1 - 0 - a_{32} - a_{23})/4 \\
h_{Z2} &= (-c_2 - a_{31} - 0 - a_{13})/4 \\
h_{Z3} &= (-c_3 - a_{21} - a_{12} - 0)/4 \\
\\
h_{R1} &= (+c_1 - 0 + a_{32} - a_{23})/4 \\
h_{R2} &= (+c_2 - a_{31} - 0 + a_{13})/4 \\
h_{R3} &= (+c_3 + a_{21} - a_{12} - 0)/4 \\
\\
h_{L1} &= (+c_1 - 0 - a_{32} + a_{23})/4 \\
h_{L2} &= (+c_2 + a_{31} - 0 - a_{13})/4 \\
h_{L3} &= (+c_3 - a_{21} + a_{12} - 0)/4
\end{aligned}
\tag{4.4}
$$

Here we have aligned Hamilton's scalar 4th parameter with Affine Geometry's homogeneous 4th coordinate, in order to represent the affine transformation matrix in terms of the `hexpe numbers`. This is the reason we place the homogeneous fourth coordinate "1" at the top of the column vector, instead of the more usual convention that often puts the extra 1 in at the bottom. In this way, we get a more natural alignment of Hamilton's 3-space $\{i, j, k\}$, with the corresponding affine space coordinates (x, y, z). Although we could have other arrangements where the quaternion scalar aligns with an affine 3-space coordinate, instead, the most meaningful representation occurs when we align the scalar with the extra affine coordinate. In this way, rotation in Hamilton's 3-space is the same kind of operation as, and corresponds with, rotation in Affine Geometry's 3-space.

SCALINGS. The first notable observation is that the M-H subalgebra $\{\boldsymbol{E}, \boldsymbol{I}_M, \boldsymbol{J}_M, \boldsymbol{K}_M\}$, now describes the shape-shifting of the 3-volume instead of 4-volume.

$$h = h_0\boldsymbol{E} + h_{M1}\boldsymbol{I}_M + h_{M2}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M \tag{4.5}$$

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} w \\ a_{11}x \\ a_{22}y \\ a_{33}z \end{pmatrix} \tag{4.6}$$

$$w'x'y'z' = a_{11}a_{22}a_{33} \cdot wxyz \tag{4.7}$$

$$x'y'z' = a_{11}a_{22}a_{33} \cdot xyz \tag{4.8}$$

Here the fourth coordinate is fixed, $w' = w = 1$, since, $a_{00} = 1$, so the nonporportional scale changes apply only to the 3-volume, $V = xyz$. When the norm of the M-H number is 1, i.e. $N_M^4 = a_{11}a_{22}a_{33} = 1$, the volume is invariant under the scale change transformation. Thus we have the same kind of volume invariance found in shape shifting the Great Pyramid. If we relaxed the requirement that the transformation be linear, and allowed the `hexpe` coefficients to be variable parameters that depend on the transforming coordinates, we could then write Khufu's Transform (2.109-111) in terms of these 4×4 matrices and corresponding `hexpe` variables,

$$\begin{pmatrix} 1 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & az+b & 0 & 0 \\ 0 & 0 & az+b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ x \\ y \\ z \end{pmatrix} \tag{4.9}$$

$$h = \frac{az+b+1}{2}\boldsymbol{E} + \frac{az+b-1}{2}\boldsymbol{K}_M \tag{4.10}$$

$$= (az+b)\frac{\boldsymbol{E}+\boldsymbol{K}_M}{2} + \frac{\boldsymbol{E}-\boldsymbol{K}_M}{2} \tag{4.11}$$

KHUFU'S TRANSFORM.

$$h = \frac{L}{H} \cdot \left(\frac{17}{15} - 2 \cdot \frac{16}{15} \cdot \frac{z}{H}\right) \frac{\boldsymbol{E}+\boldsymbol{K}_M}{2} + \frac{\boldsymbol{E}-\boldsymbol{K}_M}{2} \tag{4.12}$$

Unfortunately, this would then not be an affine transformation, for neither is the transformation "linear", nor is parallism preserved. We shall, however, return to this problem again in Projective Geometry. Now, it is a general theorem in affine geometry that,

"*Every affine transformation is equivalent to the composition of an isometry and a nonproportional scale change, or a shear with with a similarity.*"

A proproportional scaling transformation is called a homothety. The combination of homothety and isometry is called a similarity, and the isometries are rotations, inversions, reflections, translations, and their compositions. The `nonproportional scale change` referred to is pure scaling, while that described by our `hexpe numbers` can include reflections and inversions.

TRANSLATIONS. The second notable observation is that a pure 3-space translation, $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{c}$, involves all the other four 4-d `hexpe numbers`, excluding the M-H,

$$\begin{aligned} h = \boldsymbol{E} & \\ + c_1(\boldsymbol{I}_R + \boldsymbol{I}_L + \boldsymbol{I}_A - \boldsymbol{I}_Z)/4 & \\ + c_2(\boldsymbol{J}_R + \boldsymbol{J}_L + \boldsymbol{J}_A - \boldsymbol{J}_Z)/4 & \\ + c_3(\boldsymbol{K}_R + \boldsymbol{K}_L + \boldsymbol{K}_A - \boldsymbol{K}_Z)/4 & \end{aligned} \tag{4.13}$$

The translation parameters line up with the IJK axes, so c_1 becomes the coefficient of I, c_2 the coef. of J, and c_3 the coef. of K, in perfect harmony—a byproduct of our suitable labeling convention.

We could define another convenient IJK triplet to abbreviate this, say we use the subscript T for translations, then write $\boldsymbol{I}_T = (\boldsymbol{I}_R + \boldsymbol{I}_L + \boldsymbol{I}_A - \boldsymbol{I}_Z)/4$, etc..the translation would be more easily described by the number,

$$h = \boldsymbol{E} + c_1\boldsymbol{I}_T + c_2\boldsymbol{J}_T + c_3\boldsymbol{K}_T \tag{4.14}$$

However, the positive and negative units of the set of elements, $\{\boldsymbol{E}, \boldsymbol{I}_T, \boldsymbol{J}_T, \boldsymbol{K}_T\}$, do not form a group. So, this is not one of the five groups of order 8. And, in fact, this number (4.14) only has three degrees of freedom, not four. The coefficient of the unit, $\boldsymbol{E}$, is fixed at the value 1. This allows all numbers of the form, h, to have multiplicative inverses, even though the unit elements, $\boldsymbol{I}_T, \boldsymbol{J}_T, \boldsymbol{K}_T$, themselves have no inverse. Let, g, be the number obtained from flipping signs on the imaginaries,

$$g = \boldsymbol{E} - c_1\boldsymbol{I}_T - c_2\boldsymbol{J}_T - c_3\boldsymbol{K}_T \tag{4.15}$$

then it is easy to see that, g, is the inverse of h, i.e. $gh = \boldsymbol{E}$, which should be obvious also, since translation is now being represented by "products" of `hexpe numbers`. We have a special `translation subalgebra`..

$$\begin{array}{llll} \boldsymbol{E}^2 = \boldsymbol{E}, & & {\boldsymbol{I}_T}^2 = {\boldsymbol{J}_T}^2 = {\boldsymbol{K}_T}^2 = 0 \\ \boldsymbol{E}\boldsymbol{I}_T = \boldsymbol{I}_T\boldsymbol{E} = \boldsymbol{I}_T, & & \boldsymbol{I}_T\boldsymbol{J}_T = \boldsymbol{J}_T\boldsymbol{I}_T = 0 \\ \boldsymbol{E}\boldsymbol{J}_T = \boldsymbol{J}_T\boldsymbol{E} = \boldsymbol{J}_T, & & \boldsymbol{J}_T\boldsymbol{K}_T = \boldsymbol{K}_T\boldsymbol{J}_T = 0 \\ \boldsymbol{E}\boldsymbol{K}_T = \boldsymbol{K}_T\boldsymbol{E} = \boldsymbol{K}_T, & & \boldsymbol{K}_T\boldsymbol{I}_T = \boldsymbol{I}_T\boldsymbol{K}_T = 0 \end{array}$$

which results in the following relation,

$$\begin{aligned} h(\boldsymbol{c}')h(\boldsymbol{c}) &= (\boldsymbol{E} + c_1'\boldsymbol{I}_T + c_2'\boldsymbol{J}_T + c_3'\boldsymbol{K}_T) \\ &\quad \times (\boldsymbol{E} + c_1\boldsymbol{I}_T + c_2\boldsymbol{J}_T + c_3\boldsymbol{K}_T) \end{aligned} \tag{4.16}$$

$$\begin{aligned} &= (\boldsymbol{E} + (c_1' + c_1)\boldsymbol{I}_T + (c_2' + c_2)\boldsymbol{J}_T + (c_3' + c_3)\boldsymbol{K}_T) \\ &= h(\boldsymbol{c}' + \boldsymbol{c}) \end{aligned}$$

And so we have, $h(-\boldsymbol{c})h(\boldsymbol{c}) = \boldsymbol{E}$, as expected.

ROTATIONS. In Hamilton's algebra, a rotation is represented by the form, qAq^{-1}, where, q, is the quaternion describing the rotation, and, A, is the quaternion being rotated. The rotation described is a turn about in 3-space, since, if $A = A_0 + \boldsymbol{A}$, we have,

$$\begin{aligned} qAq^{-1} &= qA_0q^{-1} + q\boldsymbol{A}q^{-1} \\ &= qq^{-1}A_0 + q\boldsymbol{A}q^{-1} \\ &= A_0 + q\boldsymbol{A}q^{-1} \end{aligned} \tag{4.17}$$

where, $\boldsymbol{A}$, is the pure quaternion, $A_1i + A_2j + A_3k$. Using our pivot variable technique we can now permute the quaternions, and so, we can also write,

$$qAq^{-1} = q_RAq_R^{-1} = q_Rq_L^{-1}\hat{A} = q_L^{-1}q_R\hat{A}. \tag{4.18}$$

This means that the rotation of a vector, $\boldsymbol{v} = \boldsymbol{A}$, can be written, alternatively, $q\boldsymbol{v}q^{-1} = (q_R/q_L)\hat{\boldsymbol{v}}$. Which is to say, the ratio of "`right to left`" of a quaternion, is the operator that acts from the left to generate rotation.

The form, qAq^{-1}, closely mimics corresponding physical phenomena. A pair, $+\boldsymbol{F}$ and $-\boldsymbol{F}$, of inverse forces, acting from opposite sides of an object, produce the torque, on that object in the center, generating the physical rotation. Here, in quaternion algebra, two inverse operators, q and q^{-1}, acting from opposite sides of the variable, A, produce the transformation, on that variable in the center, which generates the rotation. The correspondence is striking, and suggests that quaternions are uniquely able to capture, in symbolic expressions, just what nature does in the real physical world. However, when it comes to manipulating algebraic expressions, it is often simpler to represent rotary operations by parameters on one side of the transforming variables.

In affine geometry, a rotation in 3-space is described by the 4×4 matrix, R, acting on the homogeneous coordinates, $\boldsymbol{x} = (w, x, y, z)$, where, $\boldsymbol{x}' = R\boldsymbol{x}$, and,

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = R \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r_{11} & r_{12} & r_{13} \\ 0 & r_{21} & r_{22} & r_{23} \\ 0 & r_{31} & r_{32} & r_{33} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{4.19}$$

with the components, r_{ij}, being constrained to pertain to rotation. This matrix has 0s in the top row and left column, so a single quaternion parameter can't represent affine rotation. Consider the R-H `hexpe` quaternion,

$$h = q_0\boldsymbol{E} + q_1\boldsymbol{I}_R + q_2\boldsymbol{J}_R + q_3\boldsymbol{K}_R \tag{4.20}$$

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = h \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

Because of the non-zeros in the top row and left column, the single quaternion operator, used for affine transformation, will include translation and scaling. In fact, if we decomposed the affine rotation matrix, R, into our hypercomplex number, this would give us,

$$\begin{aligned} 4 \cdot R = &(1 + r_{11} + r_{22} + r_{33})\boldsymbol{E} \\ &- (1 + r_{11} - r_{22} - r_{33})\boldsymbol{I}_M \\ &- (1 - r_{11} + r_{22} - r_{33})\boldsymbol{J}_M \\ &- (1 - r_{11} - r_{22} + r_{33})\boldsymbol{K}_M \\ &- (r_{32} + r_{23})(\boldsymbol{I}_A + \boldsymbol{I}_Z) \\ &- (r_{31} + r_{13})(\boldsymbol{J}_A + \boldsymbol{J}_Z) \\ &- (r_{21} + r_{12})(\boldsymbol{K}_A + \boldsymbol{K}_Z) \\ &+ (r_{32} - r_{23})(\boldsymbol{I}_R - \boldsymbol{I}_L) \\ &+ (r_{13} - r_{31})(\boldsymbol{J}_R - \boldsymbol{J}_L) \\ &+ (r_{21} - r_{12})(\boldsymbol{K}_R - \boldsymbol{K}_L) \end{aligned} \tag{4.21}$$

and we see that this involves all the `hexpe` basis elements, not just the quaternion units. But, now, remember that a rotation in Hamilton's algebra is generated from the quaternion and its inverse acting simultaneously from both sides of the variable, which we can represent by (q_R/q_L) acting from one side like the affine transformation. Consider then, the quaternion, q, that generates this same rotation in Hamilton's algebra. We have,

$$\begin{aligned} q &= q_0 + q_1 i + q_2 j + q_3 k \\ |q|^2 &= q_0^2 + q_1^2 + q_2^2 + q_3^2 \\ q_R &= q_0\boldsymbol{E} + q_1\boldsymbol{I}_R + q_2\boldsymbol{J}_R + q_3\boldsymbol{K}_R \\ q_L &= q_0\boldsymbol{E} + q_1\boldsymbol{I}_L + q_2\boldsymbol{J}_L + q_3\boldsymbol{K}_L \\ q_L^{-1} &= \frac{q_L^*}{|q|^2} = \frac{q_0\boldsymbol{E} - q_1\boldsymbol{I}_L - q_2\boldsymbol{J}_L - q_3\boldsymbol{K}_L}{|q|^2} \\ (q_R/q_L) &= q_R q_L^{-1} = \frac{q_R q_L^*}{|q|^2} \end{aligned} \tag{4.22}$$

then the product of right with left conjugate is,

$$\begin{aligned} q_R q_L^* = &q_0^2\boldsymbol{E} - q_1^2\boldsymbol{I}_M - q_2^2\boldsymbol{J}_M - q_3^2\boldsymbol{K}_M \\ &- q_2q_3(\boldsymbol{I}_A + \boldsymbol{I}_Z) - q_3q_1(\boldsymbol{J}_A + \boldsymbol{J}_Z) - q_1q_2(\boldsymbol{K}_A + \boldsymbol{K}_Z) \\ &+ q_0q_1(\boldsymbol{I}_R - \boldsymbol{I}_L) + q_0q_2(\boldsymbol{J}_R - \boldsymbol{J}_L) + q_0q_3(\boldsymbol{K}_R - \boldsymbol{K}_L) \end{aligned} \tag{4.23}$$

and equating, $R = (q_R/q_L)$, we have,

$$\begin{aligned} r_{11} &= (q_0^2 + q_1^2 - q_2^2 - q_3^2)/|q|^2 \\ r_{21} &= 2.(q_1q_2 + q_0q_3)/|q|^2 \\ r_{31} &= 2.(q_3q_1 - q_0q_2)/|q|^2 \\ r_{12} &= 2.(q_1q_2 - q_0q_3)/|q|^2 \\ r_{22} &= (q_0^2 - q_1^2 + q_2^2 - q_3^2)/|q|^2 \\ r_{32} &= 2.(q_2q_3 + q_0q_1)/|q|^2 \\ r_{13} &= 2.(q_3q_1 + q_0q_2)/|q|^2 \\ r_{23} &= 2.(q_2q_3 - q_0q_1)/|q|^2 \\ r_{33} &= (q_0^2 - q_1^2 - q_2^2 + q_3^2)/|q|^2 \end{aligned}$$

So, quaternions still represent rotations in the application of `hexpe numbers` to affine transformations, but now *it is the ratio of right to left of a quaternion that becomes the corresponding affine rotation matrix.*

NON-PROPORTIONAL SCALING. Because multiplication by a quaternion induces a rotation, the quaternion basis elements are not well suited to represent a non-proportional scaling. An attempt to represent a non-proportional scale operation using a quaternion with different scale factor coefficients, s_k, on the axes units, $q = s_0 + s_1 i + s_2 j + s_3 k$, would result in an unwanted rotation, and also produce a proportional scaling instead of a non-proportional scaling. Only the middle-hand `hexpe numbers` can represent the non-proportional scale operation. Even in Heavside-Gibbs vectors, the vector units, $\{\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$, are unable to properly represent non-proportional scaling. One couldn't use the dot product, $\boldsymbol{S} \cdot \boldsymbol{A}$, to scale the vector $\boldsymbol{A}$, by the scaling operator, $\boldsymbol{S}$, because such an operation produces a single scalar result, not a vector, and we can't compose more than one scalings, one after the other, like, $\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 \cdot \ldots \boldsymbol{S}_n \cdot \boldsymbol{A}$, because such a form has no meaning in vector algebra. With the cross product, we *can* write, $\boldsymbol{S}_1 \times \boldsymbol{S}_2 \times \ldots \boldsymbol{S}_n \times \boldsymbol{A}$, but, like in quaternions, these operations produce unwanted rotations along with the scale changes, and the scalings are again proportional. And even if we're creative, and write, $A(\boldsymbol{S}) = s_1A_1\hat{\boldsymbol{\imath}} + s_2A_2\hat{\boldsymbol{\jmath}} + s_3A_3\hat{\boldsymbol{k}}$, writing the operation component-wize, we'd still have to resort to matrix algebra to compose these scalings. Specialized scaling operations are given many names, squeezing, stretching, shrinking, expansion, contraction, compression, dilation, squashing, zooming, etc.; these are just variations of the one transformation described by the M-H hexpe numbers. There's nothing in vector algebra that will allow us to manipulate expressions involving non-proportional scalings.

SHEARING. Apart from isometries, which define the scope of transformations in `Euclidean Geometry`, there are basically two fundamentally important operations in `Affine Geometry`—the `non-proportional scaling` and the `shear` transformation. These two, together with the isometries, define everything that can be done in the affine space. A `shear` preserving x-lines has the form, $(x, y, z) \mapsto (x + \alpha y, y, z)$, where α is the shearing factor.

$$\begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \sigma \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{4.24}$$

This example of the shear shifts line segments that are parallel to the x-axis into new positions, preserving the lengths of those segments, while keeping the two perpendicular distances, y and z, from the x-axis, fixed. The `shear` and the `squeeze` alter shapes and don't preserve distances between all points. In 2-space, the squeeze transforms the square into the rectangle, while the shear changes the square into the parallelogram that shares one side with the original square. Both, however, preserve the area of the plane figure in the process. In 3-space, the cube is changed into the cuboid by the squeeze, and into a parallelopied by the shear, both preserving the volume of the shape. But, as we have discussed before, the M-H number generates the cuboid scale change, transforming aligned cubes into cuboids, and non-aligned cubes into parallelopipeds. It all depends on how the cube is aligned relative to the coordinate axes undergoing the non-proportional scaling.

$$\sigma = R(\phi) \cdot S(\lambda) \cdot R(\theta) = \tag{4.25}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\phi) & -sin(\phi) & 0 \\ 0 & sin(\phi) & cos(\phi) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta) & -sin(\theta) & 0 \\ 0 & sin(\theta) & cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\lambda_1 - \lambda_2 = \alpha \quad \lambda_1 + \lambda_2 = 1/\alpha - \alpha,$$
$$tan(\theta) = 1/\alpha, \quad \phi - \theta = (1/2 + k)\pi, \quad k \in \mathbb{Z}$$

In fact, any shear can be constructed from the composition of isometries and scalings. The particular shear, σ, given in (4.24), can be produced by a rotation, $R(\theta)$, followed by a scaling, $S(\lambda)$, followed by another rotation, $R(\phi)$, with the scaling parameters, λ_1 and λ_2, and rotation angles, θ and ϕ, all being determined by the shear factor, α, as shown in (4.25). Similarly, we can represent any nonproportional scaling by a composition of isometries, shears, and proportional scalings.

Now, we can write our shear example in the usual homogenous coordinates (4.24), or in `hexpe numbers`,

$$\begin{aligned} \sigma &= \boldsymbol{M} + \alpha \boldsymbol{K}_\sigma \\ &\text{where,} \\ \boldsymbol{M} &= \boldsymbol{E} + (\boldsymbol{I}_M + \boldsymbol{J}_M + \boldsymbol{K}_M)/2 \\ \boldsymbol{K}_\sigma &= (-\boldsymbol{K}_R + \boldsymbol{K}_L - \boldsymbol{K}_A - \boldsymbol{K}_Z)/4 \end{aligned} \tag{4.26}$$

Note that the shear factor, α, aligns with K-basis elements, even though the actual point shift is, $x' = x + \alpha y$, with no change in the z-coordinate. In fact, for the principal shears constructed from any pair of the xyz-coordinates, the shear factor always aligns with the third axis in the corresponding IJK triplet. For example,

$x' = x + \alpha y,$	$\sigma = \boldsymbol{M} + \alpha \boldsymbol{K}_\sigma$	$x' = x + \beta z,$	$\sigma = \boldsymbol{M} + \beta \boldsymbol{J}'_\sigma$
$y' = y + \alpha z,$	$\sigma = \boldsymbol{M} + \alpha \boldsymbol{I}_\sigma$	$y' = y + \beta x,$	$\sigma = \boldsymbol{M} + \beta \boldsymbol{K}'_\sigma$
$z' = z + \alpha x,$	$\sigma = \boldsymbol{M} + \alpha \boldsymbol{J}_\sigma$	$z' = z + \beta y,$	$\sigma = \boldsymbol{M} + \beta \boldsymbol{I}'_\sigma$

where $\boldsymbol{I}_\sigma \boldsymbol{J}_\sigma \boldsymbol{K}_\sigma$ refer to the combinations with the form -R+L-A-Z, and $\boldsymbol{I}'_\sigma \boldsymbol{J}'_\sigma \boldsymbol{K}'_\sigma$ have the form +R-L-A-Z instead.

STRESS & STRAIN. The alignment of the shear factor with the third axes in (4.26), indicates a twisting action, similar to the $\boldsymbol{IJ} = \boldsymbol{K}$ for rotation, and corresponds nicely with the physical `stress` and `strain` process that cause shears. The `stress` is the normal force per unit area that acts on a material body, while the `strain` is the deformation, measured by the fractional extension, that results. The ratio of stress to strain is the modulus of elasticity of the material. A high modulus material requires more effort to change its shape than a similar object with low modulus. But, there are different types of stress that can be applied to a material, causing it, for example, to stretch or shrink, compress or extend, or twist. Any given material generally has a different modulus for each type of deformation. The torsional stress produces a twist action that results in a shear, which is also called a `skew` transformation. When the torsional force has direction vector, $\boldsymbol{K}$, the deformation that occurs is in the plane at right angles to this vector, hence involves the xy-coordinates, so like in the translations, the IJK labels are in harmony here again.

PAIR PRODUCTS. Any `hexpe number`, h, can be constructed from the sum of RL pair products, that is,

$$h = A_1 B'_1 + A_2 B'_2 + \ \dots \ + A_n B'_n \tag{4.27}$$

where the A_k are R-H quaternions, and B'_k are L-H quaternions; a useful form when manipulating linear equations. This is a direct consequence of the fact that the R-L basis elements generate all other elements.

$I_R, J_R, K_R, I_L, J_L, K_L$			$I_R, J_R, K_R, I_M, J_M, K_M$		
I_R	$= +I_R$		I_R	$= +I_R$	
J_R	$= +J_R$		J_R	$= +J_R$	
K_R	$= +K_R$		K_R	$= +K_R$	
I_L	$= +I_L$		I_L	$= -I_R I_M$	$= -I_M I_R$
J_L	$= +J_L$		J_L	$= -J_R J_M$	$= -J_M J_R$
K_L	$= +K_L$		K_L	$= -K_R K_M$	$= -K_M K_R$
I_M	$= +I_R I_L$	$= +I_L I_R$	I_M	$= +I_M$	
J_M	$= +J_R J_L$	$= +J_L J_R$	J_M	$= +J_M$	
K_M	$= +K_R K_L$	$= +K_L K_R$	K_M	$= +K_M$	
I_A	$= +J_R K_L$	$= +K_L J_R$	I_A	$= -I_R K_M$	$= +K_M I_R$
J_A	$= +K_R I_L$	$= +I_L K_R$	J_A	$= -J_R I_M$	$= +I_M J_R$
K_A	$= +I_R J_L$	$= +J_L I_R$	K_A	$= -K_R J_M$	$= +J_M K_R$
I_Z	$= +K_R J_L$	$= +J_L K_R$	I_Z	$= +I_R J_M$	$= -J_M I_R$
J_Z	$= +I_R K_L$	$= +K_L I_R$	J_Z	$= +J_R K_M$	$= -K_M J_R$
K_Z	$= +J_R I_L$	$= +I_L J_R$	K_Z	$= +K_R I_M$	$= -I_M K_R$

But, we could, alternatively, represent the 15 imaginary elements in terms of any two of the five R-L-M-A-Z number types. For example, we could choose the R-H quaternions, and M-H scalings, and write all the elements in terms of RM pair products. The arbitrary hexpe number, h, in (4.27), could then be written with A_k factors being R-H quaternions, and B'_k now being M-H nonproportional scalings instead.

Projective Transformations. Once we relax the requirement that parallel lines transform into parallel lines, we move up to the field of Projective Geometry.

A `projective transformation` in 3-space is defined by three quadrilinear equations that describe how the point (x, y, z) transforms into the point (x', y', z');

$$x' = \frac{a_{10} + a_{11}x + a_{12}y + a_{13}z}{a_{00} + a_{01}x + a_{02}y + a_{03}z}$$

$$y' = \frac{a_{20} + a_{21}x + a_{22}y + a_{23}z}{a_{00} + a_{01}x + a_{02}y + a_{03}z} \quad (4.28)$$

$$z' = \frac{a_{30} + a_{31}x + a_{32}y + a_{33}z}{a_{00} + a_{01}x + a_{02}y + a_{03}z}$$

The three coordinates, (x', y', z'), are all determined by linear combinations of the initial coordinates divided by the same "scale factor", $\lambda = a_{00} + a_{01}x + a_{02}y + a_{03}z$. This scale factor is itself a linear combination of the initial coordinates, so what we have is a ratio of two multivariable linear polynomials. Composition of transformations can be represented by 4×4 matrix multiplication, since the sets of 16 a_{ij} coefficients from two successive transformations combine in a similar manner to matrix components. We can therefore treat the scale factor, λ, as a separate additional coordinate, redefine the coordinates, $x'' = \lambda x', y'' = \lambda y', z'' = \lambda z'$, add a 4th coordinate, $w'' = \lambda$, and re-write the projective transformation using 4×4 matrices,

$$\begin{pmatrix} w'' \\ x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \quad (4.29)$$

This allows us to put the rather more complicated nonlinear equation (4.28) of 3-dimensions into a much simpler linear "homogeneous" equation in 4-dimensions. Like the modification introduced in Affine Geometry, we are again able to "homogenize" the transform equations by using a simple operational trick. This time, however, the 4th coordinate has a definite interpretive meaning. It is the scale factor for the 3-space variables. The actual coordinates of the transform point being,

$$(x', y', z') = (\frac{x''}{w''}, \frac{y''}{w''}, \frac{z''}{w''}) \quad (4.30)$$

or, in terms of the point in 4-space we can write,

$$(1, x', y', z') = (\frac{w''}{w''}, \frac{x''}{w''}, \frac{y''}{w''}, \frac{z''}{w''}) \quad (4.31)$$

Initially, the 3-space point, (x, y, z), is converted into homogenous coordinates, $(w, x, y, z) = (1, x, y, z)$, with the scale factor coordinate taking the value 1. After transformation, the resulting 4-space point, (w'', x'', y'', z''), is put in the same scale as the initial coordinates by re-scaling, $(1, x', y', z')$, so that the 4th coordinate is again the unit 1. In Projective Geometry, all the 4-space points of the form, $(\lambda, \lambda x, \lambda y, \lambda z) \equiv \lambda(1, x, y, z)$, where λ is any non-zero real valued number, i.e. $\lambda \in \mathbb{R} - \{0\}$, are considered to refer to the same 3-space point, (x, y, z). So, although the transformations are described by 4×4 matrices, a matrix that induces a proportional scaling in 4-space has no significant meaning for the 3-space projective transform. Non-proportional scaling in 4-space, however, such as described by the `M-H hexpe numbers`, continue to reflect shape shifts for the 3-space.

`Affine Geometry`. In the particular special case where the scale factor is always 1, i.e. the three coefficients vanish, $a_{01} = a_{02} = a_{03} = 0$, while the fourth is the unit, $a_{00} = 1$, the Projective Transformation becomes an Affine Transformation. Affine Geometry is therefore contained within Projective Geometry, and the latter can be considered an extension of the former, where the homogeneous 4th coordinate now takes on this interpretive meaning of a transforming scale factor.

In applying our new `hexpe algebra` to describe operations in Projective Geometry, therefore, to be consistent with our corresponding approach in Affine Geometry, we once again align Hamilton's quaternion 4th parameter—the scalar—with Projective Geometry's 4th homogeneous coordinate—the scale factor. In this way, the 4×4 square matrix, $[a_{ij}]$, in equation (4.29) can be identified with that in (2.35) for the `hexpe number`.

`Projectivities`. A projective transformation is also called a "projectivity." When one of the coordinate variables is removed, say $x' \equiv x \equiv 0$, the transform equations (4.28) reduce to a pair of trilinear equations that describe the projectivity of 2-space. The yz-plane is projected onto the $y'z'$-plane. And when two of the coordinate variables are removed, say $x' \equiv x \equiv y' \equiv y \equiv 0$, these equations reduce to a single bilinear equation that describes the projectivity of 1-space—the z-axis line is projected onto itself, or onto the z'-axis.

For example, the projective transformation of this line onto itself can be written,

$$z' = \frac{\alpha z + \beta}{\gamma z + \delta} \quad (4.32)$$

where, z, is the coordinate of a point on the line, which is transformed into the point, z', by the 1-space projectivity, and the parameters, $\alpha, \beta, \gamma, \delta$, are the four coefficients that define the particular projection. This equation can be re-written in homogenous coordinates,

$$\begin{pmatrix} w'' \\ z'' \end{pmatrix} = \begin{pmatrix} \delta & \gamma \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{4.33}$$

$$(w, z) = (1, z), \qquad (1, z') = (\frac{w''}{w''}, \frac{z''}{w''}) \tag{4.34}$$

KHUFU'S TRANSFORM. Finally, we can deal with transformations of the type described by Khufu's Transform. Now consider the previous equations,

$$x' = (az + b)x \tag{2.102}$$
$$y' = (az + b)y \tag{2.103}$$
$$z' = z \tag{2.104}$$

Given that the z-coordinate is unchanged, we may replace the factor $(az + b)$ with $(az' + b)$, and re-write this transformation,

$$x' = (az' + b)x \tag{4.35}$$
$$y' = (az' + b)y \tag{4.36}$$
$$z' = z \tag{4.37}$$

then, moving this factor to the other side of the equations we write,

$$x'/(az' + b) = x \tag{4.38}$$
$$y'/(az' + b) = y \tag{4.39}$$
$$z' = z \tag{4.40}$$

This doesn't have quite the form we need to describe the transformation by a single projectivity, since one of the coordinate variables, the z-axis, isn't being scaled in the same manner as the other two, x, y. However, we may introduce a new set of intermediate coordinate variables, (x'', y'', z''), and re-write these relationships,

$$x'/(az' + b) = x'' = x \tag{4.41}$$
$$y'/(az' + b) = y'' = y \tag{4.42}$$
$$z'/(az' + b) = z'' = z/(az + b) \tag{4.43}$$

This allows us to represent Khufu's Transform by a pair of dissimilar projectivities. On the left we have a 3-space projectivity, $(x', y', z') \mapsto (x'', y'', z'')$, and on the right we have a 1-space projectivity, $z \mapsto z''$. Let us first reverse the 3-space projectivity, and write,

$$x' = x''/(-(a/b)z'' + 1/b) \tag{4.44}$$
$$y' = y''/(-(a/b)z'' + 1/b) \tag{4.45}$$
$$z' = z''/(-(a/b)z'' + 1/b) \tag{4.46}$$

This 3-space projectivity can be represented in the usual 4-dimensional homogenous coordinates,

$$\begin{pmatrix} w''' \\ x''' \\ y''' \\ z''' \end{pmatrix} = \begin{pmatrix} 1/b & 0 & 0 & -a/b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w'' \\ x'' \\ y'' \\ z'' \end{pmatrix} \tag{4.47}$$

$$(1, x', y', z') = (\frac{w'''}{w'''}, \frac{x'''}{w'''}, \frac{y'''}{w'''}, \frac{z'''}{w'''}) \tag{4.48}$$

Now, the 1-space projectivity can be represented in its usual 2-dimensional homogeneous coordinate form,

$$\begin{pmatrix} w'''' \\ z'''' \end{pmatrix} = \begin{pmatrix} b & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \tag{4.49}$$

$$(1, z'') = (\frac{w''''}{w''''}, \frac{z''''}{w''''}) \tag{4.50}$$

But, a 1-space projectivity for a line embedded in 3-space can also be written with 4×4 matrices.

$$\begin{pmatrix} w'''' \\ x'''' \\ y'''' \\ z'''' \end{pmatrix} = \begin{pmatrix} b & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \tag{4.51}$$

$$(1, x'', y'', z'') = (\frac{w''''}{w''''}, x'''', y'''', \frac{z''''}{w''''}) \tag{4.52}$$

However, the resulting 4-parameter variables are no longer "homogeneous" coordinates; and because the 4-parameter coordinates are of different types, we cannot combine the 1-space transformation with the 3-space one to form a single linear homogeneous 4-dimensional transformation. Normally, two projectivity transformations that follow each other can simply be represented by the product of the transformation matrices. But, this is only true when the projectivities are of the same type, i.e. 3-space projectivity followed by another 3-space projectivity, or a 1-space projectivity followed by another 1-space projectivity. We can just multiply the matrices and, after taking all the products of the successive transformations, we can then re-scale the column vector of the final result. In Khufu's case, however, in order to combine the two projectivities using matrix multiplication, we must re-scale the column vector between transformations. Let us define the "1-space" and "3-space" scaling operators, $S_1(X)$ and $S_3(X)$, where X is the column vector of the (w, x, y, z)-coordinates, by the following expressions,

$$S_1 \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ x \\ y \\ z/w \end{pmatrix}, \qquad S_3 \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ x/w \\ y/w \\ z/w \end{pmatrix} \tag{4.53}$$

Now let, h, be the `hexpe number` that corresponds to the 4×4 transformation matrix in (4.47), then recognizing that the 4×4 matrix in (4.51) is just the inverse of this, we can write Khufu's Transform,

$$X' = S_3(hS_1(h^{-1}X)) \tag{4.54}$$

The sequence, $S_3hS_1h^{-1}$, reading from right to left, represents: first the 1-space projectivity that scales the z-coordinate in the opposite sense to that later 3-space projectivity, so that the z-coordinate can come out of this whole operation "unchanged" in the end; this is followed immediately by a re-scaling so that the 4-parameter coordinates become true "homogeneous" coordinates for the next stage of the transformation; then the 3-space projectivity is applied, which scales the xy coordinates and simultaneously reverses the previous change in z; finally we re-scale the homogenous coordinates to bring all coordinate variables into the same reference scale we began with.

Khufu's Transform then, can be represented by a sequence of operations in Projective Geometry—this pair of linear homogeneous transformations intercepted by non-linear re-scaling—a 1-space projectivity followed by a 3-space projectivity transforms the cube into the truncated pyramid, $(x, y, z) \mapsto (x', y', z')$.

Although there are two linear transformations, only one 4×4 matrix is needed to describe Khufu's transform, the same matrix and its inverse play the operational roles in the 3-space and 1-space projective transformations, respectively. From the results (2.109-111), this matrix is,

KHUFU'S TRANSFORM:

$$h = \begin{pmatrix} \frac{15H}{17L} & 0 & 0 & \frac{32}{17H} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{4.55}$$

$$\mathcal{K} = S_3hS_1h^{-1}$$
$$\mathcal{K}\colon X \mapsto X' = \mathcal{K}X$$

This matrix transforms the cube into the truncated pyramid, keeping the volume fixed, and the height fixed.

The problem only involves various types of "scaling" operations. No other types of transformations, like translations or rotations, are required in the description. So, only that part of the projective transformation matrix that generate scaling operations have non-zero components. Scaling operations typically involve the top row and main diagonal elements of the 4×4 matrix. Translations, as in Affine Geometry, are described by the lower three components in the first column, while, rotations and shears are described by the remainder.

In the `hexpe algebra` the transformation matrix in equation (4.55) could be written,

$$h = \left(\frac{15H}{17L} - 1\right)\boldsymbol{M} + \left(\boldsymbol{E} + \frac{32}{17H}\boldsymbol{K}_S\right) \tag{4.56}$$

where,

$$\boldsymbol{M} = (\boldsymbol{E} - \boldsymbol{I}_M - \boldsymbol{J}_M - \boldsymbol{K}_M)/4$$
$$\boldsymbol{K}_S = (-\boldsymbol{K}_R - \boldsymbol{K}_L + \boldsymbol{K}_A - \boldsymbol{K}_Z)/4$$

We know from Affine Geometry that the M-H number is responsible for scaling. Thus it is understandable that a number like $\boldsymbol{M}$ should appear in this problem. However, in Projective Geometry there are additional scaling operations, not found in Affine Geometry—these are the scaling operations that enable us to transform the cube into the truncated pyramid— and these extra scale transformations are described by combinations of the other four basis elements of the `hexpe number`. The form, - R - L + A - Z, reminds us of the similar kind of form, R + L + A - Z, of the `translation subalgebra` introduced previously when discussing the affine transformation. In fact, here we can construct a corresponding "`scaling subalgebra`" to describe these extra scale operations in projectivities.

We could again define another convenient IJK triplet to abbreviate this, say we use the subscript S for scalings, then write $I_S = (-I_R - I_L + I_A - I_Z)/4$, etc..the additional scaling would be more easily described by the number,

$$h = \boldsymbol{E} + s_1\boldsymbol{I}_S + s_2\boldsymbol{J}_S + s_3\boldsymbol{K}_S \tag{4.57}$$

Note that the positive and negative units of the set of elements, $\{\boldsymbol{E}, \boldsymbol{I}_S, \boldsymbol{J}_S, \boldsymbol{K}_S\}$, do not form a group. So again, this is not one of the five groups of order 8. And, in fact, this number (4.57) only has three degrees of freedom, not four. The coefficient of the unit, $\boldsymbol{E}$, is fixed at the value 1. This allows all numbers of the form, h, to have multiplicative inverses, even though the unit elements, $\boldsymbol{I}_S, \boldsymbol{J}_S, \boldsymbol{K}_S$, themselves have no inverse.

Let, g, be the number obtained from flipping the signs on the imaginary parts,

$$g = \boldsymbol{E} - s_1\boldsymbol{I}_S - s_2\boldsymbol{J}_S - s_3\boldsymbol{K}_S \tag{4.58}$$

then it is easy to see that, g, is the inverse of h,

$$gh = \boldsymbol{E} \tag{4.59}$$

which should be obvious also, since the transpose of the 4×4 transformation matrix exchanges the translation parameters with these scaling parameters, so this "projective scaling" component of the transform is similar in nature to the translation, thus a parallel analysis follows.

Apart from projectivities that map n-space onto n-space, there are projective transformations that map n-space onto (n-k)-space. For example, when the four coefficients that determine z' in (4.28) vanish, i.e. $a_{30} = a_{31} = a_{32} = a_{33} = 0$, we obtain a projection of 3-space onto 2-space; the 3-space points are mapped onto the plane, $(x, y, z) \mapsto (x', y')$. This is the more typical type of projective transformation found in general use, since artists use this to render drawings and paintings, computer graphic designers use this to render 3-d images on the display, cameras and movie projectors effectively transform the world into flat screen representations, the human eye captures the objective world's 3-d images on the 2-d surface of the retina, and so on.

`Perspective Transformations`. A special type of projectivity, called "perspective projection" or "perspective transformation," or simply a "perspectivity," deals with the way the human eye actually perceives objects in the three dimensional world. The main feature being that objects farther away appear smaller in size than similar objects nearer to the observer. When this natural vision's scaling of size with distance is incorporated into the transformation we call it a perspective projection[18].

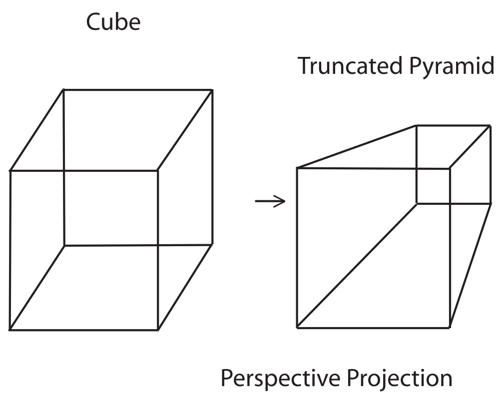

FIG. 5: The Eye's Perspective

Not only the sizes of objects, but the sizes of different parts of the same object are distorted by perspective, the nearer face of a cube appearing larger than the face farther away. This particular scaling makes the cube take on the appearance of a truncated pyramid.

After discovering the hidden cube in the Great Pyramid, and reflecting on the fact that, for a cube to appear to be a truncated pyramid with that particular orientation, the cube must be hovering in the sky above the observer, and he must be looking upwards at it, the Projective Geometer is struck by the apparent significance: *someone used these giant limestone blocks to scrawl a message in the desert that says* `''look up !''`

Spacetime Transformations. Apart from being useful in the study of 3-space, hexpe numbers can also find application in spacetime 4-space. Here the fourth quaternion parameter is interpreted as time. There is, in fact, a fundamental connection between scaling and time in physics. If we heat a gas, it expands. That expansion isn't instantaneous, however, it takes time. We have a scale change when the gas expands. All the molecules are on average further apart. Change in size is often a measure of the passage of time. Astronomers use the expansion of the universe to measure time from the big bang. Biological organisms also tend to increase in size with their age—the newly fertilized cell, verses the adult phenotype, are differentiated by manifest complexity and size. So, there's an intricate connection among the concepts of time, heat, and scale changes, and these are all naturally represented by the same 4th parameter of the quaternion variables when modeling phenomena.

We shall see later in this section, that when we interpret the 4th quaternion coordinate as time, the Electric field then takes on a corresponding 4th parameter, which we call the Temporal Field. This can be identified with that particular *reversible heat* found in Thermoelectric phenomena, again illustrating the `time-heat-scaling` connection. There are two kinds of heat, *irreversible heat* and *reversible heat*, and there are two types of scale changes, those described by the main diagonal components and those described by the top row components of the projective transformation matrix.

PARITY & CHIRALITY. At the June 1845 meeting of the *British Association for the Advancement of Science*, Hamilton is reported to have requested that the following conjecture of his be placed on the records[19]:

"Is there not an analogy between the fundamental pair of equations ij=k ji=-k, and the facts of opposite currents of electricity corresponding to opposite rotations?"

Hamilton realized that quaternions encapsulate the handed character of 3-space particularly well. Thus, he felt that his geometric algebra would accurately describe the essential polarity found in electric phenomena, that depends on this right-hand verses left-hand distinction. When we simultaneously invert all three space coordinates, we also convert the handedness of shapes and forms in geometry. This operation is called "`Parity`", and given the symbol, $P\colon (x, y, z) \mapsto (-x, -y, -z)$. [20]

Lord Kelvin introduced the term "`Chiral`" for handed geometric forms. A geometrical figure has "`Chirality`" if "*its image in a plane mirror, ideally realized, cannot be brought to coincide with itself.*"[21]

If we're given a right-hand form, there are two basically different operations that convert it into the left-hand. We can invert just one coordinate, e.g. $(x, y, z) \mapsto (-x, y, z)$, like the plane mirror, or three, as in Parity. But, an inversion of two coordinates simultaneously, e.g. $(x, y, z) \mapsto (-x, -y, z)$, does not change the hand. Is there a way to distinguish these two left-hands?

Pseudo Left verses True Left. Which is the real left hand? This interesting question arises once we realize that, ${\boldsymbol{I}_R}^* \neq \boldsymbol{I}_L$, and therefore, $i_R^* \neq i_L$.

In Hamilton's quaternions, we usually consider multiplication by the conjugate of a quaternion to represent rotation in the opposite sense, which it does, and therefore consider this a left hand rotation, since the basis elements for quaternions are by definition right handed. This creates the feeling that the facility for including left hand actions is already there in his algebra, and a separate left hand basis is not needed to express such actions.

Indeed, there are two ways to indicate left hand actions. One way is to take the conjugate, the other way is to multiply from the other side of a variable. Thus, if $A \cdot B$ is considered a right hand action by virtue of the way A operates on B, then $B \cdot A$ is automatically the reverse situation, and is now a left hand action by virtue of the way A is now operating from the opposite side of B.

So, Hamilton's algebra already has two different ways of expressing the left hand, since, $B \cdot A \neq A \cdot B^*$. And, for certain, we can use his quaternion algebra to express both right and left actions in the equations that model phenomena. Let's consider an example, to see what's involved in using right and left combinations.

With ordinary quaternion variables we have two ways to write a pair product, AB or BA. Since each of these products contains an implied rotary movement, one containing a `right-turn` while the other a `left-turn`, we automatically include a bias when writing down expressions with one or the other form of the pair product. If we're trying to model some physical phenomena that doesn't have such a bias inherent in the actual process being represented, we then need to compensate for this automatic bias by combining the two forms of products.

So, we define the symmetric product, $\{A, B\}$, and the anti-symmetric product, $[A, B]$, to provide alternate ways to write down quaternion expressions;

$$\begin{aligned} \{A, B\} &= 1/2 \cdot (AB + BA) \\ [A, B] &= 1/2 \cdot (AB - BA) \end{aligned} \tag{4.60}$$

If we're only dealing with ordinary quaternion variables, then the four forms, $AB, BA, \{A, B\}, [A, B]$, as defined above, should suffice to provide enough flexibility to enable our model constructions. But when dealing with operators we have to resolve yet another ambiguity. In what direction does the operator act? Say, now, $\boldsymbol{A}$ is the operator, and B is an ordinary quaternion variable. Does $B \cdot \boldsymbol{A}$ mean $B \leftarrow \boldsymbol{A}$ or $B \cdot \boldsymbol{A} \rightarrow$? We have to clarify which direction the operator acts. So we modify our definitions accordingly, with right and left arrows,

$$\begin{aligned} \{\boldsymbol{A}, B\} &= 1/2 \cdot (\boldsymbol{A} \rightarrow B + B \leftarrow \boldsymbol{A}) \\ [\boldsymbol{A}, B] &= 1/2 \cdot (\boldsymbol{A} \rightarrow B - B \leftarrow \boldsymbol{A}) \end{aligned} \tag{4.61}$$

to clarify that the operator $\boldsymbol{A}$ is acting on B regardless of where the operator sits. Our arrows, $\rightarrow$ and $\leftarrow$, resolve that operator ambiguity. Now, it is possible to engage in more complicated constructions than this, but we shall not find the need to do so.

Alexander McAulay [1] , in his *Utility of Quaternions in Physics* [22] [2], suggested [3] a somewhat more comprehensive notation than ours. A product consisting of any number of quaternion parameters, $PQRBHS\boldsymbol{A}ZW$, could be dis-ambiguated by adding a subscript to the operator [4] and variable to indicate which variable it was intended to act upon. Thus, $PQRB_1HS\boldsymbol{A}_1ZW$, would indicate that, despite its apparent position in the string of variables, the operator $\boldsymbol{A}$ acts only on the one variable B, all other parameters being variables that compose the multi-product through their quaternion type multiplication, but otherwise are not affected by the operator.

McAulay's level of detail is beyond our current requirements. Let us now turn to our example of combining right and left actions in a practical application.

In a previous paper[23] [1] we showed how to write Maxwell Equations in Hamilton's quaternions. We shall highlight some of the results here for our illustration.

The main idea behind that paper is that by extending the concept of right actions and left actions from ordinary products `to operators`, we have right acting operators and left acting operators, $D \rightarrow A$ and $A \leftarrow D$, to consider, which must now both be included in the expressions used to model physical phenomena.

We start with the hypothesis that spacetime is a quaternion structure, so that we can write the event variables, (ct, x, y, z), in the form, $r = ct + xi + yj + zk$. Then, any four-vector (quaternion) variable, that depends on these events, r, can be written,

$$\mathcal{A} = \mathcal{A}_0 1 + \mathcal{A}_1 i + \mathcal{A}_2 j + \mathcal{A}_3 k \tag{4.62}$$

Now, it doesn't matter what this variable, $\mathcal{A}$, represents at this point. It could be any given quantity. All we know is that it has four component parameters corresponding to the four degrees of freedom in our spacetime, and that it is a function of those spacetime points, $\mathcal{A} = \mathcal{A}(r)$. But, we are now told that this quantity is not a constant, it changes, fluctuates with time and space, and undergoes various types of modifications all describable by continuous functions.

So, then we ask—how do we best describe the changes, the fluctuations, and modifications, of our quantity, $\mathcal{A}$?

First method of attack, is to consider first order differential changes. We know that a certain operator describes the fastest rate of change of a dependent variable, to first order approximation,

$$\frac{d}{dr} = \frac{\partial}{\partial ct}1 + \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k \qquad (4.63)$$

So, to describe the changes, we evaluate the action of this operator, d/dr, on the given quantity, $\mathcal{A}$.

There's only one problem now. We're in the land of quaternions, and there are two ways to employ this operator.

$$\frac{d}{dr} \rightarrow \mathcal{A} \qquad = \texttt{or} = \qquad \mathcal{A} \leftarrow \frac{d}{dr} \qquad (4.64)$$

If we let the operator act towards the right, $d/dr \rightarrow \mathcal{A}$, we will include a right-hand rotary movement in the description of the fluctuation we see. If we let the operator act towards the left, $\mathcal{A} \leftarrow d/dr$, we will include a left-hand rotary movement in the description of the fluctuation we see. Well, what are we looking for? To a certain extent, the answers we get back depend on the questions we decide to ask. What if we're told that the phenomena being described has no inherent rotary movements, that the fluctuations are all scale changes, expanding and contracting, shearing and shifting, squeezing and stretching, and so on. What's the best question to ask then?

$$\{\frac{d}{dr}, \mathcal{A}\} \qquad = \texttt{or} = \qquad [\frac{d}{dr}, \mathcal{A}] \qquad (4.65)$$

We could examine how the quantity changes by using the symmetric derivative, $\{d/dr, \mathcal{A}\}$. Or, suppose we're told that there are no such scale changes, the quantity is changing entirely by rotary fluctuations? We might use the anti-symmetric derivative, $[d/dr, \mathcal{A}]$, in this case. So, one or the other of the two alternative product expressions might be more appropriate for our description of the changes. It's really a matter of our viewpoint.

So then, someone comes along and says, look, these left and right derivatives mix up too many transformation types at the same time. I don't know much about the physical phenomena under investigation. Not sure whether it has fluctuations based on rotary changes or scale changes, or whatever. I need a viewpoint that separates these mixed up transformation types as much as possible. Give me the component of the fluctuation that looks the same when viewed in either a right-hand frame or a left-hand frame, because then I'll know I've eleminated any hand-dependent features. And give me the component of the fluctuation that looks the most different between these right and left coordinate frames, because then I'll know what depends on the hand.

$$(4.66)$$

$$\begin{aligned} \mathcal{E} &= -\{d/dr, \mathcal{A}\} = -1/2(d/dr \rightarrow \mathcal{A} + \mathcal{A} \leftarrow d/dr) \\ \mathcal{B} &= +[d/dr, \mathcal{A}] \;\; = +1/2(d/dr \rightarrow \mathcal{A} - \mathcal{A} \leftarrow d/dr) \end{aligned}$$

Well, as it turns out, that's the symmetric and anti-symmetric derivatives. Let's call them, $\mathcal{E}$ and $\mathcal{B}$. We choose to define $\mathcal{E}$ with an extra minus sign $'-'$, just because it turns out that, owing to the way the classical electric field has been defined historically, these then become the electric and magnetic fields when the quantity, $\mathcal{A}$, under investigation, is the quaternion version of the electromagnetic potential.

But there is nothing inherently electromagnetic about this discussion so far. These are simply first order changes of a quantity, $\mathcal{A}$, broken down into the views of hand-independent and hand-dependent components.

The quaternion version of the `Maxwell Equations` can then be written as the following pair of equations,

$$[d/dr, \mathcal{B}] = +\{d/dr, \mathcal{E}\} \qquad (4.67)$$

$$[d/dr, \mathcal{E}] = -\{d/dr, \mathcal{B}\} \qquad (4.68)$$

These are the homogeneous equations that describe how the fields change when there are no source charges nor sources currents present.

The second of these equations can be easily shown to be an algebraic identity, when given the definitions of $\mathcal{E}$ and $\mathcal{B}$ in (4.66), owing to the fact that quaternions are `associative`, and thus the order of differentiation doesn't matter—differentiating from the right, before differentiating from the left, produces the same result as differentiating from the left, before differentiating from the right; $d/dr \rightarrow (\mathcal{A} \leftarrow d/dr) = (d/dr \rightarrow \mathcal{A}) \leftarrow d/dr$.

It is in the first of these equations where we have our first contact with real physics. Now we have something that we can call electromagnetic. It's a property that tells us how those components of the first order fluctuations themselves change relative to each other.

Where there are no source charges nor source currents, the hand-dependent component of the change in the magnetic field, is always the same as the hand-independent component of the change in the electric field.

Expressing these quaternion variables in terms of the usual Heaviside-Gibbs vectors, $\mathcal{A} = (U, \vec{\boldsymbol{A}})$, $\mathcal{B} = (0, \vec{\boldsymbol{B}})$, $\mathcal{E} = (T, \vec{\boldsymbol{E}})$, we obtain the following vector equations,

$$\begin{aligned} curl(\vec{\boldsymbol{B}}) &= +1/c \cdot \partial\vec{\boldsymbol{E}}/\partial t + grad(T) \\ curl(\vec{\boldsymbol{E}}) &= -1/c \cdot \partial\vec{\boldsymbol{B}}/\partial t \\ div(\vec{\boldsymbol{E}}) &= +1/c \cdot \partial T/\partial t \\ div(\vec{\boldsymbol{B}}) &= 0 \end{aligned} \qquad (4.69)$$

$$\begin{aligned} T &= -1/c \cdot \partial U/\partial t + div(\vec{\boldsymbol{A}}) \\ \vec{\boldsymbol{E}} &= -grad(U) - 1/c \cdot \partial\vec{\boldsymbol{A}}/\partial t \\ \vec{\boldsymbol{B}} &= curl(\vec{\boldsymbol{A}}) \end{aligned} \qquad (4.70)$$

Now there's one twist that is introduced with the quaternion approach. While the space components of the quaternion parameters, $\mathcal{E}$ and $\mathcal{B}$, are identical with the classical vector fields, $\vec{\boldsymbol{E}}$ and $\vec{\boldsymbol{B}}$, the quaternion electric field has an extra time component, not present in classical electromagnetism. We call this scalar field component the `temporal field`, T. So, what we really have is, $\mathcal{E} = (T, \vec{\boldsymbol{E}})$, and, $\mathcal{B} = (0, \vec{\boldsymbol{B}})$. This is one of the essential contributions of this quaternion method to the theory—a new field component. We shall see, however, that we still get the classical electromagnetism to fall out of this analysis, despite the novel field.

When we add the charge density, $4\pi\rho$, and current density, $4\pi\vec{\boldsymbol{J}}/c$, terms, in their usual places, the equations become,

$$\begin{aligned} curl(\vec{\boldsymbol{B}}) &= +1/c \cdot \partial\vec{\boldsymbol{E}}/\partial t + grad(T) + 4\pi\vec{\boldsymbol{J}}/c \\ curl(\vec{\boldsymbol{E}}) &= -1/c \cdot \partial\vec{\boldsymbol{B}}/\partial t \\ div(\vec{\boldsymbol{E}}) &= +1/c \cdot \partial T/\partial t + 4\pi\rho \\ div(\vec{\boldsymbol{B}}) &= 0 \end{aligned} \tag{4.71}$$

These are the inhomogeneous electromagnetic equations. They are very much like the classical equations, except we've got, in addition, two new terms in the equations, $grad(T)$ and $1/c \cdot \partial T/\partial t$, resulting from the new T-field, which the classical equations don't have.

But, we can consider the homogenous equations, without those charge and current sources, and call the solutions to these equations, $\vec{\boldsymbol{E}}_T$ and $\vec{\boldsymbol{B}}_T$, where we've added the subscript, $_T$, to indicate these fields result from the new `homogenous equations` with the two terms, $1/c \cdot \partial T/\partial t$ and $grad(T)$, effectively playing the role of charge density and current density, respectively.

Then we put the real charge and current densities back into the equations, to obtain the expressions for the fields with the usual source terms present.

Now we define, $\vec{\boldsymbol{D}} = \vec{\boldsymbol{E}} - \vec{\boldsymbol{E}}_T$, and $\vec{\boldsymbol{H}} = \vec{\boldsymbol{B}} - \vec{\boldsymbol{B}}_T$, our equations become,

$$\begin{aligned} curl(\vec{\boldsymbol{H}}) &= +1/c \cdot \partial\vec{\boldsymbol{D}}/\partial t + 4\pi\vec{\boldsymbol{J}}/c \\ curl(\vec{\boldsymbol{E}}) &= -1/c \cdot \partial\vec{\boldsymbol{B}}/\partial t \\ div(\vec{\boldsymbol{D}}) &= 4\pi\rho \\ div(\vec{\boldsymbol{B}}) &= 0 \end{aligned} \tag{4.72}$$

Thus we obtain the general media inhomogeneous equations for `Maxwell Equations`, with the electric displacement vector, $\vec{\boldsymbol{D}}$, and magnetic field vector, $\vec{\boldsymbol{H}}$, absorbing the two new terms implicated by the quaternion approach. The temporal quantities, $1/c \cdot \partial T/\partial t$ and $grad(T)$, are then obscured by the concepts of *polarization* and *magnetization* that describe the media.

So, this example shows, by combining right and left actions, Hamilton's algebra can indeed be used to construct the expressions for practical models that describe phenomena. Solving these quaternion equations are another matter. That type of difficulty is what prompted the construction of vector algebra.

But, we didn't need to resort to things like complexified quaternions, or biquaternions, or modify the definition of quaternions so the basis elements have positive squares, or invent biquats, and the like, all of which previous authors have employed in their attempt to find the right way to construct Maxwell's Equations from quaternions. Hamilton's quaternions, in their native form, are more than adequate to do the job, once we recognize that `both left acting operators and right acting operators are required`.

It's a non-abelian algebra. One needs to get out of the frame of mind of an abelian algebraist to recognize the significance of left and right combinations.

But now we come back to the central issue of the two left hands. We've used the reversal of the pair product, i.e. AB verses BA, to express the left hand action in our construction of the model for the electromagnetic equations. And we've made no use of the conjugate.

Our left hand quaternion element, $\boldsymbol{I}_L$, is derived from this same concept of reversing the pair product, in an attempt to find ways to solve such dual hand problems. So, what is the difference between a left hand quaternion and the conjugate of a right hand quaternion? In other words, how exactly is $\boldsymbol{I}_R{}^*$ different from $\boldsymbol{I}_L$? Note the difference, $\boldsymbol{I}_R{}^*\boldsymbol{I}_R = \boldsymbol{E}$ but $\boldsymbol{I}_L\boldsymbol{I}_R = \boldsymbol{I}_M$. *A left hand quaternion is not the same thing as the conjugate of a right hand quaternion!*

Note that, $\boldsymbol{I}_M\colon (w, x, y, z) \mapsto (-w, -x, y, z)$. So, look what happens to the coordinate of I,

$$\boldsymbol{I}_R{}^*\boldsymbol{I}_R\colon x \mapsto +x$$
$$\boldsymbol{I}_L\boldsymbol{I}_R\colon x \mapsto -x$$

Both these operators, $\boldsymbol{I}_R{}^*$ and $\boldsymbol{I}_L$, reverse the rotary movement introduced by $\boldsymbol{I}_R$, but the left hand element results in an additional inversion of the coordinate pair, $w \to -w$ and $x \to -x$. *The true left hand operator combines with the true right hand operator to create an inversion in the remaining two axes, while leaving the plane of the rotary movement in a net unchanged state.* The combination of left and right cancels the rotation, but at the same time induces a reflection with that very same rotary plane acting as the mirror.

$\boldsymbol{I}_M$ is a mirror operator, it induces a reflection in the imaginary plane perpendicular to its axis, and at the same time also causes an inversion of the scalar.

With our new left hand basis elements we can now re-write the equations (4.66) that define the electric and magnetic fields,

$$\hat{\mathcal{E}} = -1/2 \cdot \left(\frac{d}{dr_R} + \frac{d}{dr_L}\right)\hat{\mathcal{A}} \quad (4.73)$$

$$\hat{\mathcal{B}} = +1/2 \cdot \left(\frac{d}{dr_R} - \frac{d}{dr_L}\right)\hat{\mathcal{A}} \quad (4.74)$$

with the definitions of the right and left derivative operators,

$$\frac{d}{dr_R} = \frac{\partial}{\partial ct}1 + \frac{\partial}{\partial x}i_R + \frac{\partial}{\partial y}j_R + \frac{\partial}{\partial z}k_R \quad (4.75)$$

$$\frac{d}{dr_L} = \frac{\partial}{\partial ct}1 + \frac{\partial}{\partial x}i_L + \frac{\partial}{\partial y}j_L + \frac{\partial}{\partial z}k_L \quad (4.76)$$

The pair of `Maxwell Equations` (4.67-68) can then also be re-written,

$$\left(\frac{d}{dr_R} - \frac{d}{dr_L}\right)\hat{\mathcal{B}} = +\left(\frac{d}{dr_R} + \frac{d}{dr_L}\right)\hat{\mathcal{E}} \quad (4.77)$$

$$\left(\frac{d}{dr_R} - \frac{d}{dr_L}\right)\hat{\mathcal{E}} = -\left(\frac{d}{dr_R} + \frac{d}{dr_L}\right)\hat{\mathcal{B}} \quad (4.78)$$

with the operators now all on one side of the variables. We could combine these two equations by adding and subtracting one from the other to obtain the alternative pair,

$$\frac{d}{dr_L}\hat{\mathcal{B}} = -\frac{d}{dr_R}\hat{\mathcal{E}} \quad (4.79)$$

$$\frac{d}{dr_R}\hat{\mathcal{B}} = +\frac{d}{dr_L}\hat{\mathcal{E}} \quad (4.80)$$

Then, because we now know the product rules for the basis elements, $i_R i_L = i_L i_R = i_M$, $i_R j_L = k_A$, $j_R i_L = k_Z$, etc.., we have,

$$\begin{aligned} \frac{d}{dr_R}\frac{d}{dr_L} &= \left(\frac{\partial}{\partial ct}\right)^2 1 \\ &+ \left(\frac{\partial}{\partial x}\right)^2 i_M + \left(\frac{\partial}{\partial y}\right)^2 j_M + \left(\frac{\partial}{\partial z}\right)^2 k_M \\ &+ \frac{\partial^2}{\partial y\partial z}i_A + \frac{\partial^2}{\partial z\partial x}j_A + \frac{\partial^2}{\partial x\partial y}k_A \\ &+ \frac{\partial^2}{\partial z\partial y}i_Z + \frac{\partial^2}{\partial x\partial z}j_Z + \frac{\partial^2}{\partial y\partial x}k_Z \\ &+ \frac{\partial}{\partial ct}\left(\frac{\partial}{\partial x}i_R + \frac{\partial}{\partial y}j_R + \frac{\partial}{\partial z}k_R\right) \\ &+ \frac{\partial}{\partial ct}\left(\frac{\partial}{\partial x}i_L + \frac{\partial}{\partial y}j_L + \frac{\partial}{\partial z}k_L\right) \\ &= \frac{d}{dr_L}\frac{d}{dr_R} \end{aligned} \quad (4.81)$$

and so the left and right derivative operators commute with each other. We can then differentiate the left and right sides of the equations (4.79-80) to obtain,

$$\frac{d}{dr_L}\frac{d}{dr_L}\hat{\mathcal{B}} = -\frac{d}{dr_L}\frac{d}{dr_R}\hat{\mathcal{E}} \quad (4.82)$$

$$\frac{d}{dr_R}\frac{d}{dr_R}\hat{\mathcal{B}} = +\frac{d}{dr_R}\frac{d}{dr_L}\hat{\mathcal{E}} \quad (4.83)$$

Adding these equations, and applying the commuting result from (4.81), we have,

$$\left(\left(\frac{d}{dr_R}\right)^2 + \left(\frac{d}{dr_L}\right)^2\right)\hat{\mathcal{B}} = 0 \quad (4.84)$$

In a similar way, we obtain the corresponding result for the electric field,

$$\left(\left(\frac{d}{dr_R}\right)^2 + \left(\frac{d}{dr_L}\right)^2\right)\hat{\mathcal{E}} = 0 \quad (4.85)$$

Now lets call this combined operator that sums the squares of right and left derivative operators, Ω, except we also normalize with an extra factor of $1/2$, so,

$$\Omega = \frac{1}{2}\left(\left(\frac{d}{dr_R}\right)^2 + \left(\frac{d}{dr_L}\right)^2\right) \quad (4.86)$$

Then our homogenous electromagnetic equations just become, $\Omega\hat{\mathcal{E}} = 0$, and, $\Omega\hat{\mathcal{B}} = 0$.

If we started with the definitions of the electric and magnetic fields given in (4.73-74), and simply replaced the fields by these definitions in equations (4.77-78), we'd find that the electromagnetic potential also obeys this same form of equation, $\Omega\hat{\mathcal{A}} = 0$. Here omega is,

$$\begin{aligned} \Omega &= \left(\frac{\partial}{\partial ct}\right)^2 - \left(\frac{\partial}{\partial x}\right)^2 - \left(\frac{\partial}{\partial y}\right)^2 - \left(\frac{\partial}{\partial z}\right)^2 \\ &+ \frac{\partial}{\partial ct}\left(\frac{\partial}{\partial x}i_R + \frac{\partial}{\partial y}j_R + \frac{\partial}{\partial z}k_R\right) \\ &+ \frac{\partial}{\partial ct}\left(\frac{\partial}{\partial x}i_L + \frac{\partial}{\partial y}j_L + \frac{\partial}{\partial z}k_L\right) \end{aligned} \quad (4.87)$$

When there are sources, the inhomogeneous equation for the electromagnetic potential becomes[24],

$$\Omega\hat{\mathcal{A}} = 4\pi\hat{\mathcal{J}} \quad (4.88)$$

with the quaternion current source, $\hat{\mathcal{J}} = (\rho, \vec{\boldsymbol{J}}/c)$.

`True Left`: Now there are two aspects to the concept of the left hand. There is the left hand of dynamic geometry, which is manifest through rotations. Then, there's the left hand of static geometry, which is manifest through forms. A left hand molecule, for example, can be turned into a right hand molecule, using a mirror. But, that same left hand molecule cannot be transformed into a right hand molecule using either a right hand rotation or a left hand rotation. The $\boldsymbol{I}_L$ is a true complement to $\boldsymbol{I}_R$ in that they then combine to provide for both mirror and rotary operations that flip the hand of orientation.

5. FORMAL DEFINITIONS.

Rules of Decomposition. The following rules enable us to construct our particular flavor of hypercomplex numbers from square matrices defined on $\mathbb{R}$.

(1)—To decompose an $N \times N$ square matrix, we construct a set of exactly N^2 linearly independent basis matrices, to keep the number of degrees of freedom the same in the new hypercomplex number. (2)—For each basis matrix, B, every column and every row has exactly one non-zero component, which is either $+1$ or -1, all other matrix components are 0. (3)—If E is the unit matrix, every basis matrix, B, in the set, must have $B^2 = \pm E$; thus $\mathtt{det}(B) = \pm 1$, since these are matrices over the reals. (4)—If A and B are in the set, then either, $+AB$, or, $-AB$, is in the set.

We call our new `square matrix decomposition hypercomplex numbers`—`hypermats`. The extended quaternion we call `hexpe number` is then an example of a 4^2`-dim hypermat number`, while the extended complex number we called `alternating complex number` is an example of a 2^2`-dim hypermat number`. For an $N \times N$ matrix there are $2^{N-1} \times N!$ ways to construct a potential basis matrix, from which we need to find the right N^2 of them. It is likely that N must be a power of 2 for successful decomposition.

HEXPENTAQUATERNIONS. We define the set of 16 basis elements $\mathbb{X}_b^+ = \{e, i_P, j_P, k_P : \ \text{P} = \text{R,L,M,A,Z}\ \}$, according to the following product rules,

$$e^2 = e,$$
$$i_R^2 = j_R^2 = k_R^2 = -e, \quad i_L^2 = j_L^2 = k_L^2 = -e,$$

$$\forall\ u = i, j, k;\ \mathtt{P} = \mathtt{R}, \mathtt{L}$$
$$e u_P = u_P e = u_P,$$

$$\forall\ u, v, w = i, j, k;\ \mathtt{P}, \mathtt{Q}, \mathtt{S} = \mathtt{R}, \mathtt{L}$$
$$u_P(v_Q w_S) = (u_P v_Q) w_S,$$

$$j_R k_R = -k_R j_R = i_R, \qquad k_L j_L = -j_L k_L = i_L,$$
$$k_R i_R = -i_R k_R = j_R, \qquad i_L k_L = -k_L i_L = j_L,$$
$$i_R j_R = -i_R j_R = k_R, \qquad j_L i_L = -i_L j_L = k_L,$$

$$j_R k_L = \ k_L j_R = i_A, \qquad k_R j_L = \ j_L k_R = i_Z,$$
$$k_R i_L = \ i_L k_R = j_A, \qquad i_R k_L = \ k_L i_R = j_Z,$$
$$i_R j_L = \ j_L i_R = k_A, \qquad j_R i_L = \ i_L j_R = k_Z,$$

$$i_R i_L = i_L i_R = i_M,$$
$$j_R j_L = j_L j_R = j_M,$$
$$k_R k_L = k_L k_R = k_M,$$

Since this system is generated by the R-H and L-H basis elements, these are all the rules required to establish the remaining product laws. From these defining rules, using the associative product law given, we construct the following additional derived rules[25].

$$i_M^2 = j_M^2 = k_M^2 = e,$$
$$i_A^2 = j_A^2 = k_A^2 = e, \quad i_Z^2 = j_Z^2 = k_Z^2 = e,$$

$$\forall\ u = i, j, k;\ \mathtt{P} = \mathtt{M}, \mathtt{A}, \mathtt{Z}$$
$$e u_P = u_P e = u_P,$$

$$\forall\ u, v, w = i, j, k;\ \mathtt{P}, \mathtt{Q}, \mathtt{S} = \mathtt{R}, \mathtt{L}, \mathtt{M}, \mathtt{A}, \mathtt{Z}$$
$$u_P(v_Q w_S) = (u_P v_Q) w_S,$$

$$j_M k_M = k_M j_M = -i_M,$$
$$k_M i_M = i_M k_M = -j_M,$$
$$i_M j_M = j_M i_M = -k_M,$$

$$j_A k_A = k_A j_A = -i_A, \qquad k_Z j_Z = j_Z k_Z = -i_Z,$$
$$k_A i_A = i_A k_A = -j_A, \qquad i_Z k_Z = k_Z i_Z = -j_Z,$$
$$i_A j_A = i_A j_A = -k_A, \qquad j_Z i_Z = i_Z j_Z = -k_Z,$$

In addition to this set of 16 elements, we define the complementary set of their negative values, $\mathbb{X}_b^- = \{-e, -i_P, -j_P, -k_P : \ \text{P} = \text{R,L,M,A,Z}\ \}$.

These two together form the group of order 32, called the `Hexpentaquaternion Group`: $\mathbb{X}_b = \mathbb{X}_b^+ \cup \mathbb{X}_b^-$.

The abbreviation `Hexpe` may be used for convenience. Accordingly, we define an `Hexpe Number` as the linear combination of these basis elements with real valued coefficients,

$$\begin{aligned} h \ =\ & h_0 e \\ & + \ h_{R1} i_R + h_{R2} j_R + h_{R3} k_R \\ & + \ h_{L1} i_L + h_{L2} j_L + h_{L3} k_L \\ & + \ h_{M1} i_M + h_{M2} j_M + h_{M3} k_M \\ & + \ h_{A1} i_A + h_{A2} j_A + h_{A3} k_A \\ & + \ h_{Z1} i_Z + h_{Z2} j_Z + h_{Z3} k_Z \end{aligned}$$

$$h_P \in \mathbb{R}$$

The symbol $\mathbb{X}_n$ represents the set of all such `hexpe numbers`, and includes the special number 0. With the two operators of addition, $+$, and multiplication, $\cdot$, this set forms the `Hexpe Algebra`. If the context is clear whether reference is being made to `the group` of basis elements, or `the algebra` of hypercomplex numbers built on this group of elements, the symbol $\mathbb{X}$, without the subscript, maybe used to refer to either, i.e. $\mathbb{X}_b$ or $\mathbb{X}_n$.

Use of the multiplication symbol, $\cdot$, is optional, it being understood that simple juxtaposition represents the same multiplication operation, i.e. $gh \equiv g \cdot h$.

The element, e, is called the `scalar` element, and may be replaced by the real unit, 1, it being understood that, for any $\lambda \in \mathbb{R}$, we have, $\lambda \cdot e \equiv \lambda \cdot 1 \equiv \lambda$.

Let, e_k, $k = 0, 1, 2, ...15$, be the sixteen basis elements, $\{e, i_R, j_R, ...j_Z, k_Z\}$, with the scalar element, $e_0 = e$, and the remaining 15 imaginary elements arranged in any particular fixed order. Then, any three `hexpe numbers`, h, g, f, may be written,

$$h = \sum_{k=0}^{15} h_k e_k, \quad g = \sum_{k=0}^{15} g_k e_k, \quad f = \sum_{k=0}^{15} f_k e_k$$

where the coefficients, $h_k, g_k, f_k, \in \mathbb{R}$. Then, the rules of addition and multiplication are,

$$h + g = \sum_{k=0}^{15} (h_k + g_k) e_k, \quad h \cdot g = \sum_{k=0}^{15} \sum_{j=0}^{15} h_k g_j e_k e_j$$

and we have the following, `Closure, Commutativity, Identity, Associative` rules,

$$\forall\ h, g, f \ \in \mathbb{X}_n$$

$$\left\| \begin{array}{l} h + g \ \in \mathbb{X}_n \\ h + g = g + h \\ h + 0 = 0 + h = h \\ (h + g) + f = h + (g + f) \end{array} \right\| \begin{array}{l} h \cdot g \in \mathbb{X}_n \\ h \cdot g \neq g \cdot h \\ h \cdot e = e \cdot h = h \\ h \cdot (g \cdot f) = (h \cdot g) \cdot f \end{array}$$

`and Distributive Laws;`

$$h \cdot (g + f) = (h \cdot g) + (h \cdot f)$$
$$(h + g) \cdot f = (h \cdot f) + (g \cdot f)$$

The commutative law for multiplication does not hold generally, i.e. $h \cdot g \neq g \cdot h$, but there are subsets of `hexpe numbers` that commute, i.e. $h \cdot g = g \cdot h$; and there are subsets that anti-commute, i.e. $h \cdot g = -g \cdot h$. So the $\neq$ sign here means "`not always`" rather than "`never`."

`Inverses` exist for the + operator, but not always for the $\cdot$ operator. For addition, the inverse of, h, is written, $-h$, and accordingly, the auxiliary operation, $-$, of `subtraction`, is defined,

$$\begin{aligned} h + (-h) &= 0 \\ h + (-g) &\equiv h - g \end{aligned}$$

The multiplicative inverse of, h, is written, h^{-1}, and is given by the general formula

$$h \cdot h^{-1} = h^{-1} \cdot h = e$$

$$\begin{aligned} h^{-1} = &\ (w_0 e \\ &+ w_{R1} i_R + w_{R2} j_R + w_{R3} k_R \\ &+ w_{L1} i_L + w_{L2} j_L + w_{L3} k_L \\ &+ w_{M1} i_M + w_{M2} j_M + w_{M3} k_M \\ &+ w_{A1} i_A + w_{A2} j_A + w_{A3} k_A \\ &+ w_{Z1} i_Z + w_{Z2} j_Z + w_{Z3} k_Z)/d \end{aligned}$$

$$w_P \in \mathbb{R}$$

The weight factors, w_P, are given by the cubic form,

$$w_P = h_P^3 - h_P \sum_\alpha s_{P,\alpha} h_\alpha^2 - 2 \sum_{\alpha\beta\gamma} s_{P,\alpha\beta\gamma} h_\alpha h_\beta h_\gamma$$

where, $P \in \{0, R1, ..., Z3\}$, and the normalizing determinant factor is,

$$d = h_0 \cdot w_0 - \sum_{k=1,2,3} (h_{Rk} \cdot w_{Rk} + h_{Lk} \cdot w_{Lk})$$
$$+ \sum_{k=1,2,3} (h_{Mk} \cdot w_{Mk} + h_{Ak} \cdot w_{Ak} + h_{Zk} \cdot w_{Zk})$$

The signs, $s_{P,\alpha}$ and $s_{P,\alpha\beta\gamma}$, and ranges α, β, γ, are given by the expanded forms for w_P in (TABLE T.3-IV).

When the multiplicative inverse for a number, g, exists, the auxiliary operations, / and \, i.e. forward slash and backslash, for `division from the right` and `division from the left`, are defined,

$$h/g = h \cdot g^{-1}, \qquad g \backslash h = g^{-1} \cdot h$$

For convenience also, the fraction variants of these divisions are defined with the two, `\dashv` and `\vdash`, symbols, placed to the right and left of the denominator expression,

$$h/g = \frac{h}{g \dashv}, \qquad g \backslash h = \frac{h}{\vdash g}$$

and, when available, two variations of `\frac`, where the horizontal bar combines with these symbols, to form right and left horizontal bars, —⊣ and ⊢—, may be used. The suggested names for these fraction commands are, `\rfrac` and `\lfrac`.

R,L,M,A,Z These symbols may be referred to by the names: RIGHT, LEFT, META, ALPHA, ZETA. The latter three being collectively referred to as MIDDLE-HAND numbers—the term 'HAND" being generally extended to apply now to any particular 4-d hypercomplex sub-algebra of the `hexpe number` system.

PAIR PRODUCTS. Any arbitrary `hexpe number`, h, can be written as the sum of RL pair products,

$$h = A_1 B_1' + A_2 B_2' + A_3 B_3' + \ldots + A_n B_n'$$

where, A_k are R-H quaternions, and B_k' are L-H quaternions.

PIVOT VARIABLES. Let Hamilton's right hand quaternions be the set $\mathbb{H}_R \subset \mathbb{X}_n$, and left hand quaternions be the set $\mathbb{H}_L \subset \mathbb{X}_n$. The `conjugate` of a parameter, $B \in \mathbb{H}_R$ `or` $B \in \mathbb{H}_L$, is obtained by reversing the signs on the imaginary units of the number, and is formally written, B^*. A `hand transformation` on a parameter, B, is one that changes the basis elements from right hand to left hand, or left hand to right hand, while leaving the coefficients unchanged. This change is much like the taking the conjugate, except here the real left hand replaces the right hand, instead of the pseudo left constructed by sign changes. We write B^H, to indicate this hand change.

$$(B_0 + B_1 i_R + B_2 j_R + B_3 k_R)^* = (B_0 - B_1 i_R - B_2 j_R - B_3 k_R)$$
$$(B_0 + B_1 i_R + B_2 j_R + B_3 k_R)^H = (B_0 + B_1 i_L + B_2 j_L + B_3 k_L)$$
$$(B_0 + B_1 i_L + B_2 j_L + B_3 k_L)^* = (B_0 - B_1 i_L - B_2 j_L - B_3 k_L)$$
$$(B_0 + B_1 i_L + B_2 j_L + B_3 k_L)^H = (B_0 + B_1 i_R + B_2 j_R + B_3 k_R)$$

We also use the descriptive subscripts R-L to clarify which basis is being employed to accompany the same set of coefficients, $\{B_0, B_1, B_2, B_3\}$, for a given parameter, B, under consideration. Then, $B_L = (B_R)^H$, and, $B_R = (B_L)^H$. It being understood that, for a given B,

$$\begin{aligned} B_R &\equiv (B_0 + B_1 i_R + B_2 j_R + B_3 k_R) \\ B_L &\equiv (B_0 + B_1 i_L + B_2 j_L + B_3 k_L) \end{aligned}$$

Now let the parameters, $q, B \in \mathbb{H}_R$, then the product, qB, may be written, $B^H \hat{q}$, where $B^H \in \mathbb{H}_L$, and $\hat{q}$ is called a "`pivot variable`" represented by the caret ˆ placed on top of the parameter.

Then, for products involving '`pivot variables`' the following `Commutative`, `Associative`, and `Distributive` laws hold,

$$\begin{aligned} qB &= B^H \hat{q} \\ A(B^H \hat{q}) &= (AB^H)\hat{q} \\ G\hat{q} + F\hat{q} &= (G+F)\hat{q} \\ H(G\hat{q} + F\hat{p}) &= (HG)\hat{q} + (HF)\hat{p} \end{aligned}$$

`where`

$H, G, F \in \mathbb{X}_n$ `and`

$q, p, A, B \in \mathbb{H}_R \quad , \quad B^H \in \mathbb{H}_L$

$=$ `or` $=$

$q, p, A, B \in \mathbb{H}_L \quad , \quad B^H \in \mathbb{H}_R$

From these four laws, we conclude the following special cases,

$$A(qB) = A(B^H \hat{q}) = (AB^H)\hat{q}$$

$$\begin{aligned} Aq &= A\hat{q}, \quad \texttt{when } B = 1 \\ q &= \hat{q}, \quad \texttt{when } A = B = 1 \end{aligned}$$

That is to say, if all factors are already on the L.H.S of the variable, the q may be automatically promoted to a pivot, $\hat{q}$; and again, if neither left nor right factors are present on a variable, q, that variable may be immediately promoted to a pivot, $\hat{q}$. But, when a non-trivial factor (i.e. $B \notin \mathbb{R}$) is present on the R.H.S of the variable, q, that variable may only be promoted to a pivot by moving the factor over to the L.H.S where the factor then changes its hand.

There are two distributive laws involving pivots—a right-distributive law, and a left-distributive law of significant difference. The first defines how a quaternion pivot variable on the right distributes over a sum of `hexpe number` parameters, the second defines how an `hexpe` number factor on the left distributes over a sum of quaternion pivots themselves also containing L.H.S. `hexpe` factors. This latter distributive law results in an important special case, when, $F = -1$, and, $\hat{p} = \hat{c}$, i.e. the second quaternion, p, is the usual inhomogeneous parameter, c, in the linear equation,

$$H(G\hat{q} - \hat{c}) = (HG)\hat{q} - H\hat{c}$$

During the manipulation of algebraic expressions, parameters in the expression may be converted to and from pivot equivalents at any time during the process of reckoning. Thus some terms in the expression may contain pivots, while others contain the related variables in the original state. However, an expression involving a mix of variables and pivot equivalent variables may only employ a distributive law to aggregate and further simplify the expression iff either all the relevant variables are in pivot format or all the relevant variables are in the original native format.

The most general linear quaternion equation may be written,

$$A_1 q B_1 + A_2 q B_2 + ... + A_n q B_n - C = 0$$

where either, $A_k, B_k, C, q \in \mathbb{H}_R$, or, $A_k, B_k, C, q \in \mathbb{H}_L$. But, in either case, the equation may be re-arranged using the given laws thus,

$$\begin{aligned} A_1(qB_1) + A_2 q B_2 + ... + A_n q B_n - C &= 0 \\ A_1(B_1^H \hat{q}) + A_2 q B_2 + ... + A_n q B_n - C &= 0 \\ (A_1 B_1^H)\hat{q} + A_2 q B_2 + ... + A_n q B_n - C &= 0 \\ (A_1 B_1^H)\hat{q} + A_2(qB_2) + ... + A_n q B_n - C &= 0 \\ (A_1 B_1^H)\hat{q} + A_2(B_2^H \hat{q}) + ... + A_n q B_n - C &= 0 \\ (A_1 B_1^H)\hat{q} + (A_2 B_2^H)\hat{q} + ... + A_n q B_n - C &= 0 \\ ((A_1 B_1^H) + (A_2 B_2^H))\hat{q} + ... + A_n q B_n - C &= 0 \\ &\vdots \\ ((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))\hat{q} - C &= 0 \end{aligned}$$

To proceed to the next step, we must convert the inhomogeneous parameter, C, to pivot format, in order to apply the second distributive law, which is automatically involved when we multiply the equation throughout by some chosen parameter. In this case, we need to multiply by the inverse of $\hat{q}$'s L.H.S factor, to reduce this term to just $\hat{q}$. Given that any quaternion that has only trivial factors on its left and right may be promoted immediately at any time to pivot, we may write,

$$((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))\hat{q} - \hat{C} \;= 0$$

Now we may multiply by that inverse,

$$((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))^{-1} \cdot$$
$$[((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))\hat{q} - \hat{C}] = 0$$

and apply the "`second distributive law`" for pivots, which yields,

$$\hat{q} - ((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))^{-1}\hat{C} = 0$$

Given that the pivot variable, $\hat{q}$, stands all by itself, with neither left nor right factor parameters, we may immediately convert this back to the original quaternion format, q, and thus now write this equation,

$$q - ((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))^{-1}\hat{C} = 0$$

In general, if this inverse exists, it may be written in the form,

$$((A_1 B_1^H) + (A_2 B_2^H) + ... + (A_n B_n^H))^{-1} =$$
$$P_1 Q_1^H + P_2 Q_2^H + ... + P_m Q_m^H$$

where the parameters, P_k, Q_k, are of the same hand-orientation as the given parameters, A_k, B_k, and thus, Q_k^H, are the alternate hand versions of the Q_k. This then lets us write the equation,

$$q - ((P_1 Q_1^H) + (P_2 Q_2^H) + ... + (P_m Q_m^H))\hat{C} = 0$$

and continue to simplify, using the first distributive law for pivots,

$$q - (P_1 Q_1^H)\hat{C} - ((P_2 Q_2^H) + ... + (P_m Q_m^H))\hat{C} = 0$$
$$q - P_1(Q_1^H \hat{C}) - ((P_2 Q_2^H) + ... + (P_m Q_m^H))\hat{C} = 0$$
$$q - P_1(C Q_1) - ((P_2 Q_2^H) + ... + (P_m Q_m^H))\hat{C} = 0$$
$$q - P_1 C Q_1 - ((P_2 Q_2^H) + ... + (P_m Q_m^H))\hat{C} = 0$$
$$\vdots$$
$$q - P_1 C Q_1 - P_2 C Q_2 - ... - P_m C Q_m = 0$$

finally, moving the known quantities to the R.H.S of the equation, we can present the solution,

$$q = P_1 C Q_1 + P_2 C Q_2 + ... + P_m C Q_m$$

where all the parameters are once again in either Hamilton's right hand quaternions, or all in Hamilton's left hand quaternions, depending on which system the original linear problem is specified.

ALGEBRA OF THE SPLIT OPERATOR

The concept of the pivot variable is useful in the solution of linear equations. However, this method requires the pivot always stand to the right of the expressions, and so it is difficult to extend this technique to polynomials of degree higher than one. Accordingly, we may also define corresponding expressions, to those in this split representation technique, using the alternative split operator method, where we introduce, $\cdot$ and $\otimes$, the two multiplication operators, to now replace the single product. This then becomes the starting point for research in further generalizations of this algebra to solve equations of higher degree. The methods developed so far in this direction are incomplete, and discussion is beyond the scope of this paper. Here we only deal with the linear quaternion equation, and this is adequately solved with the pivot method.

6. CONCLUSIONS.

Algebra of Geometry. When Hamilton went looking for his "triplets," the buzz and excitement of the times was all focused on finding an extension to the complex number that could facilitate `'rotations'` in 3-space the way the complex number so efficiently handled rotations in the plane. With a little more prodding, mathematicians of the time would have found that similar number extensions could also express 3-space transformations other than just rotations. After all, rotations are only one part of the general transformation of a space object. But for some reason, the fixation was on rotations, and after the discovery of quaternions the fascination faded.

The search went off in the direction of finding higher dimensional numbers with the restrictive, but useful, `'square norm'` property, rather than seeking other numbers that might provide alternative ways to view and efficiently express other transformations in the already established `space geometry`. Yet, Hamilton was seeking to develop an *algebra of geometry*—the regular geometry of three dimensions which was already known through its cartesian coordinate formulation. This was his impulse. Well, that geometry contains more than just translations, rotations, and proportional scale changes. There are `nonproportional scale` changes also to consider, which can't be easily expressed with Hamilton's numbers.

It is interesting, however, that by looking for a way to include the left-hand quaternions which Hamilton neglected, we are then automatically led to also include those very additional number extensions which in fact do enable us to construct a more complete geometric algebra for the 3-space.

Quaternion Equations. One of the major challenges in quaternion algebra is how to efficiently solve quaternion equations. Simple polynomial equations are rather difficult, but even the linear equation requires the use of additional methods over that of regular commutative algebra. Here, we must generally resort to matrix algebra to find our solutions. With the introduction of our new `hexpe algebra`, and its corresponding pivot variable operational techniques, we now have an alternative method to solve the most general linear equations of the type,

$$A_1qB_1 + A_2qB_2 + ... + A_nqB_n = C \tag{2.1}$$

being able to easily express the solution in the form,

$$q = P_1CQ_1 + P_2CQ_2 + ... + P_mCQ_m \tag{2.221}$$

using the sort of elementary algebra procedures that are very familiar in commutative algebra. So, even though non-abelian algebra presents a challenge, for the general linear equation in one variable, at least, we are able to reduce the work of reckoning in the discovery of solutions to a type no more excessive than regular algebra usually requires.

One can then extend these ideas further to solve `systems of linear equations` in more than one quaternion variable. For example, a pair of linear equations in two unknown quaternion variables, p and q,

$$A_{11}pB_{11} + A_{12}qB_{12} = C_1 \tag{6.1}$$
$$A_{21}pB_{21} + A_{22}qB_{22} = C_2 \tag{6.2}$$

where all parameters, $As, Bs, Cs, p, q,$ are right hand quaternions, could be re-written,

$$A_{11}B'_{11}\hat{p} + A_{12}B'_{12}\hat{q} = \hat{C}_1 \tag{6.3}$$
$$A_{21}B'_{21}\hat{p} + A_{22}B'_{22}\hat{q} = \hat{C}_2 \tag{6.4}$$

with the B parameters converted into their left hand B' counterparts. Then, if the coefficients of the $\hat{q}$ have inverses, we may multiply by these to obtain,

$$(A_{12}B'_{12})^{-1}A_{11}B'_{11}\hat{p} + \hat{q} = (A_{12}B'_{12})^{-1}\hat{C}_1 \tag{6.5}$$
$$(A_{22}B'_{22})^{-1}A_{21}B'_{21}\hat{p} + \hat{q} = (A_{22}B'_{22})^{-1}\hat{C}_2 \tag{6.6}$$

then solving for, $\hat{p}$, we get,

$$\hat{p} = \frac{(A_{12}B'_{12})^{-1}\hat{C}_1 - (A_{22}B'_{22})^{-1}\hat{C}_2}{\vdash((A_{12}B'_{12})^{-1}A_{11}B'_{11} - (A_{22}B'_{22})^{-1}A_{21}B'_{21})} \tag{6.7}$$

similarly, if coefs of $\hat{p}$ have inverses, solving for, $\hat{q}$, we get,

$$\hat{q} = \frac{(A_{11}B'_{11})^{-1}\hat{C}_1 - (A_{21}B'_{21})^{-1}\hat{C}_2}{\vdash((A_{11}B'_{11})^{-1}A_{12}B'_{12} - (A_{21}B'_{21})^{-1}A_{22}B'_{22})} \tag{6.8}$$

The problem can now be expressed in the usual matrix form,

$$\begin{pmatrix} A_{11}B'_{11} & A_{12}B'_{12} \\ A_{21}B'_{21} & A_{22}B'_{22} \end{pmatrix} \begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} \tag{6.9}$$

with the recognition that the inverse of a matrix with non-abelian components is a bit more complicated,

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_1} & 0 \\ 0 & \frac{1}{D_2} \end{pmatrix} \begin{pmatrix} (A_{12}B'_{12})^{-1} & -(A_{22}B'_{22})^{-1} \\ (A_{11}B'_{11})^{-1} & -(A_{21}B'_{21})^{-1} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} \tag{6.10}$$

our simple overall determinant factor, in the usual matrix algebra, being replaced here with a diagonal matrix of different dividing factors,

$$\begin{aligned} D_1 &= (A_{12}B'_{12})^{-1}A_{11}B'_{11} - (A_{22}B'_{22})^{-1}A_{21}B'_{21} \\ D_2 &= (A_{11}B'_{11})^{-1}A_{12}B'_{12} - (A_{21}B'_{21})^{-1}A_{22}B'_{22} \end{aligned} \tag{6.11}$$

and the non-commuting nature of the parameters prevents us from simplifying without further information on the makeup of each parameter. But such an inverse can, nevertheless, always be expressed in the general form[26],

$$\begin{pmatrix} A_{11}B'_{11} & A_{12}B'_{12} \\ A_{21}B'_{21} & A_{22}B'_{22} \end{pmatrix}^{-1} = \begin{pmatrix} P_{11p}Q'_{p11} & P_{12q}Q'_{q12} \\ P_{21r}Q'_{r21} & P_{22s}Q'_{s22} \end{pmatrix} \tag{6.12}$$

when it exists; where the Ps are right hand quaternions, and the $Q's$ are left hand quaternions. Then, we can write (6.10) in the form,

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} P_{11p}Q'_{p11} & P_{12q}Q'_{q12} \\ P_{21r}Q'_{r21} & P_{22s}Q'_{s22} \end{pmatrix} \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} \tag{6.13}$$

applying matrix product rules, this gives,

$$\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix} = \begin{pmatrix} P_{11p}Q'_{p11}\hat{C}_1 + P_{12q}Q'_{q12}\hat{C}_2 \\ P_{21r}Q'_{r21}\hat{C}_1 + P_{22s}Q'_{s22}\hat{C}_2 \end{pmatrix} \tag{6.14}$$

and then finally, we move the left hand quaternions, i.e. $Q's$, over to the right of the inhomogeneous parameters, where they change into right hand quaternions, i.e. Qs, letting us remove the carets, and we have,

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} P_{11p}C_1Q_{p11} + P_{12q}C_2Q_{q12} \\ P_{21r}C_1Q_{r21} + P_{22s}C_2Q_{s22} \end{pmatrix} \tag{6.15}$$

which is our solution for the pair of quaternions, p and q, everything being expressed once again entirely in the right hand quaternion system.

The ease with which this technique lets us think and work with familiar constructs cannot be denied—the power of the method lies in its essential simplicity. The `hexpe algebra`, even if it were only used to solve such quaternion problems, would be worthy of study.

Extending Numbers. Hamilton was only looking for numbers with three parameters, written either, (x, y, z) or $x + yi + zj$, to describe 3-space rotations. But after 10 years (1833-1843)[27][1] struggling with the concept of how to appropriately multiply these triplets, he finally got his breakthrough when he realized he needed just one more imaginary parameter, k. He could then solve this problem, with numbers of the form, $q = w + xi + yj + zk$, and the additional insight that he needed to relax the commutative law for products, $ij = -ji = k, ki = -ik = j, jk = -kj = i$. Our challenge to find the extension to Hamilton's quaternions that would allow the inclusion of both left and right hands in the same algebra is a little less dramatic, but does have one thing in common with Hamilton's discovery—we also find we need more imaginary parameters than we initially sought. The seven basis elements $\{1, i_R, j_R, k_R, i_L, j_L, k_L\}$, need to be augmented by a further nine imaginary units, $\{i_M, j_M, k_M, i_A, j_A, k_A, i_Z, j_Z, k_Z\}$, to complete the algebra. Moreover, one needs to relax the requirement that every number have a multiplicative inverse, and settle instead for sub-domains where inverses are defined which are themselves bounded by other sub-domains where they are not.

These domains where the inverse fails to exist are not randomly placed about in the space, however, but fall on well defined planes that define sensible limits of physical transformations. The middle hand numbers generate nonproportional scale changes, without any mathematical restriction on the signs of the scale factors. Scale factors can be positive or negative. Thus they can include inversions or reflections accompanying the pure magnitude change. Consider the archetypal middle-hand transformation generated by the M-H number;

$$h_M: \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} w' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} a_{00}w \\ a_{11}x \\ a_{22}y \\ a_{33}z \end{pmatrix} \tag{6.16}$$

Here the scale factors, $a_{00}, a_{11}, a_{22}, a_{33}$, are the same four factors that appear in the denominator of the number's inverse formula. The norm, $N_M^4 = a_{00}a_{11}a_{22}a_{33}$, vanishes when one or more of these scale factors are zero. That's the only time the inverse fails to exist. But, these are the times when the the continuous changing of a scale factor would take it from the positive to negative sign, resulting in an inversion or reflection of the coordinate undergoing transformation. Now, a continuous change cannot take a physical object into its mirror image. So what these planes do, is divide the region of the 4-d hypercomplex space up into spaces where continuous change of the parameters result in physically possible transformations, and what is impossible physically is also prevented mathematically by requiring a discontinuous jump across one of these planes.

This then allows us to treat the discontinuous planes the way complex analysis treats branch cuts and singular points, and to develop an analytic calculus for the `hexpe algebra` to the extent possible within the constraints of non-abelian products. So, although the `hexpe algebra` appears formidable with its many degrees of freedom and times of failing inverses, it really doesn't add much more complexity to that already existing under Hamilton's non-commuting quaternion calculus. And, as we have seen, the `hexpe algebra` actually helps to solve some problems that Hamilton's algebra could only propose but not then find solutions for without the outside help of something like matrix algebra lending a helping hand.

HEXADECANIONS. Now there is already another previously established favorite hypercomplex system with sixteen degrees of freedom. That system is called

`Hexadecanions`[28]. These numbers are obtained from a pair of octonions through a process called the `Cayley-Dickson` construction for pair products. The octonions are themselves constructed from a pair of quaternions using this same process. The quaternions are constructed from complex numbers this way again; the complex numbers constructed from the reals.

CAYLEY-DICKSON PROCESS[29]: $(A,B)^* \equiv (A^*, -B)$

$$(A,B)(C,D) = (AC - DB^*, A^*D + CB) \tag{6.17}$$

The Cayley-Dickson process is a method of "doubling" that creates new algebras from existing ones, by using a pair of numbers from a lower dimensional hypercomplex algebra. This whole idea of multiplying such doublets originates with Hamilton, who introduces the concept in the 1830s to represent complex numbers by ordered pairs (x, y), equivalent to the then usual $x + iy$ form of the number. Cayley and Dickson then develop the idea for the particular "normed" algebras that appeared on the scene soon after Hamilton's discovery of quaternions opened up the whole idea of extending the number system to higher and wilder entertaining entities.

Hamilton's good friend John Graves discovers the eight dimensional hypercomplex variety soon after the announcement of quaternions. He chooses to call these numbers `Octaves`, and gives Hamilton the task of publicizing the discovery for him. Hamilton, however, forgets to do this, and Cayley independently discovers and publishes the same system 15 months later, gets the credit, and the 8-d variety comes to be called either Cayley numbers or Octonions.

From the Cayley-Dickson construction, we can see that the `hexadecanions` are built out of `four right-hand` quaternions. The `hexpentaquaternions`, by contrast, are built from `a pair of left and right` quaternions, which generate an additional triplet of commutative 4-d hypercomplex numbers in the process of constructing the algebra. The hexpe construction process is derived from matrix algebra, rather than erected out of thin air with a special pair product construction like the Cayley-Dickson algebras. So, in one sense, the `hexpe algebra` is not new. It is equivalent to an old and already established matrix algebra[30]. It is only a new way of looking and working with an old familiar subject. It is a `re-interpretation` of an existing algebra, demonstrating that the old algebra is equivalent to an hypercomplex number with basis elements being the square roots of $+1$ and -1, just like the other hypercomplex algebras in existence, and showing that left and right quaternions combined together in one system were already playing a role, hidden from view, behind the scenes in every transformation matrix.

In the `hexpe algebra` left and right quaternions commute with each other, even though left hand quaternions do not commute among themselves, neither do right hand quaternions commute among themselves.

$$A_R B_L = B_L A_R, \qquad A_R \in \mathbb{H}_R,\ B_L \in \mathbb{H}_L \tag{6.18}$$

If we found other ways to combine the left hand and right hand quaternions into one system, we should not expect this commuting property to continue to hold.

The Cayley-Dickson process is a beautiful system, simple, and elegant. But there is one problem with the whole idea. It continually stacks up right hand upon right hand, erecting a geometrically unbalanced structure that denies the reality before our very eyes—that the universe has right hand AND left hand components. Everywhere we look, our experience instructs us that physical phenomena possesses this dual nature of left-hand and right-hand in combination. The question then is, can we adopt the beautiful construction of Cayley-Dickson and find a way to balance it, so that we can better reflect the true dual handedness of our actual physical space? [31]

`Octivos` (6.19)

$$\begin{aligned}
1:&\ (A,B)(C,D) = (AC - B^H C, A^H D - BD)\\
2:&\ (A,B)(C,D) = (AC - (BD)^H, A^H D + C^H B)\\
3:&\ (A,B)(C,D) = (AC - (B^*D)^H, (A^*)^H D + C^H B)\\
4:&\ (A,B)(C,D) = (AC - (D^*B)^H, D^*A^H + C^H B)\\
&\qquad A, C \in \mathbb{H}_R, \quad B, D \in \mathbb{H}_L
\end{aligned}$$

OCTIVOS. Instead of pairing up two right hand quaternions, therefore, we consider doublet products with one right and one left hand number in the pair. Say then that we have, $(A,B) \equiv (A_R, B_L)$, $A_R \in \mathbb{H}_R$, $B_L \in \mathbb{H}_L$, etc.. we'd like to define the product, $(A,B)(C,D) = (E,F)$, so that, $(E,F) \equiv (E_R, F_L)$, $E_R \in \mathbb{H}_R$, $F_L \in \mathbb{H}_L$. If we just put this pair into the existing Cayley-Dickson formula, we'd end up with a mix of left and right quaternions in each half of the resultant doublet. But then we recall that the conjugate is really a way to construct a pseudo left hand from a right hand quaternion, so we think that maybe we can just replace this operation with the true hand transformation operator, $BD^* \mapsto BD^H$ and $A^*D \mapsto A^H D$. That doesn't quite work out either, but with a little thought, remembering that $(BD)^H = D^H B^H$, and realizing that we'd also maybe like to have, $(A,0)(C,0) = (AC,0)$, so that the product of two R-H quaternions result in another R-H, and Similarly, $(0,B)(0,D) = s(0,BD)$, the product of two L-H quaternions should result in a L-H, with the possible exception of some overall sign, s, factoring in all these deliberations, we eventually end up with the four candidate definitions in (6.19). We shall call these numbers "`octivos`"[32]. The `octivo algebras` are also non-associative and non-commutative just like the octonions. We leave it as an exercise for the reader to review the octivo product tables (TABLE T.4), explore these numbers, and decide if they are interesting numbers to study.

TABLE T.1 ⊢

TABLE OF 16 MATRIX BASIS ELEMENTS

$$I_R = J_RK_R = -K_RJ_R \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}, \quad J_R = K_RI_R = -I_RK_R \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K_R = I_RJ_R = -J_RI_R \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix},$$

$$I_L = K_LJ_L = -J_LK_L \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_L = I_LK_L = -K_LI_L \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix}, \quad K_L = J_LI_L = -I_LJ_L \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}$$

$$I_A = J_RK_L = +K_LJ_R \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_A = K_RI_L = +I_LK_R \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K_A = I_RJ_L = +J_LI_R \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix},$$

$$I_Z = K_RJ_L = +J_LK_R \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad J_Z = I_RK_L = +K_LI_R \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad K_Z = J_RI_L = +I_LJ_R \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$E = -I_RI_R = -I_LI_L \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad I_M = I_RI_L = +I_LI_R \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix},$$

$$J_M = J_RJ_L = +J_LJ_R \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}, \quad K_M = K_RK_L = +K_LK_R \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The products of the four elements $\{I_R, J_R, I_L, J_L\}$ generate all the other members in this set of sixteen matrices, and under the binary operation of matrix multiplication, this set, together with their negatives, then form a representation of the particular Group of Order 32 called the HEXPENTAQUATERNION GROUP: $\mathbb{X}_b$.

THE HEXPENTAQUATERNION GROUP, $\mathbb{X}_b$

TABLE T.2 ‖

$\times$	E	I_A	J_A	K_A	I_R	J_R	K_R	I_M	J_M	K_M	I_L	J_L	K_L	I_Z	J_Z	K_Z
E	E	I_A	J_A	K_A	I_R	J_R	K_R	I_M	J_M	K_M	I_L	J_L	K_L	I_Z	J_Z	K_Z
I_A	I_A	E	$-K_A$	$-J_A$	$-K_M$	$-K_L$	J_Z	I_Z	$-I_L$	$-I_R$	$-J_M$	K_Z	$-J_R$	I_M	K_R	J_L
J_A	J_A	$-K_A$	E	$-I_A$	K_Z	$-I_M$	$-I_L$	$-J_R$	J_Z	$-J_L$	$-K_R$	$-K_M$	I_Z	K_L	J_M	I_R
K_A	K_A	$-J_A$	$-I_A$	E	$-J_L$	I_Z	$-J_M$	$-K_L$	$-K_R$	K_Z	J_Z	$-I_R$	$-I_M$	J_R	I_L	K_M
I_R	I_R	K_M	$-K_Z$	$-J_L$	$-E$	K_R	$-J_R$	$-I_L$	I_Z	$-I_A$	I_M	K_A	J_Z	$-J_M$	$-K_L$	J_A
J_R	J_R	$-K_L$	I_M	$-I_Z$	$-K_R$	$-E$	I_R	$-J_A$	$-J_L$	J_Z	K_Z	J_M	I_A	K_A	$-K_M$	$-I_L$
K_R	K_R	$-J_Z$	$-I_L$	J_M	J_R	$-I_R$	$-E$	K_Z	$-K_A$	$-K_L$	J_A	I_Z	K_M	$-J_L$	I_A	$-I_M$
I_M	I_M	I_Z	J_R	K_L	$-I_L$	J_A	$-K_Z$	E	$-K_M$	$-J_M$	$-I_R$	$-J_Z$	K_A	I_A	$-J_L$	$-K_R$
J_M	J_M	I_L	J_Z	K_R	$-I_Z$	$-J_L$	K_A	$-K_M$	E	$-I_M$	I_A	$-J_R$	$-K_Z$	$-I_R$	J_A	$-K_L$
K_M	K_M	I_R	J_L	K_Z	I_A	$-J_Z$	$-K_L$	$-J_M$	$-I_M$	E	$-I_Z$	J_A	$-K_R$	$-I_L$	$-J_R$	K_A
I_L	I_L	J_M	$-K_R$	$-J_Z$	I_M	K_Z	J_A	$-I_R$	$-I_A$	I_Z	$-E$	$-K_L$	J_L	$-K_M$	K_A	$-J_R$
J_L	J_L	$-K_Z$	K_M	$-I_R$	K_A	J_M	I_Z	J_Z	$-J_R$	$-J_A$	K_L	$-E$	$-I_L$	$-K_R$	$-I_M$	I_A
K_L	K_L	$-J_R$	$-I_Z$	I_M	J_Z	I_A	K_M	$-K_A$	K_Z	$-K_R$	$-J_L$	I_L	$-E$	J_A	$-I_R$	$-J_M$
I_Z	I_Z	I_M	$-K_L$	$-J_R$	J_M	$-K_A$	$-J_L$	I_A	I_R	I_L	K_M	$-K_R$	$-J_A$	E	$-K_Z$	$-J_Z$
J_Z	J_Z	$-K_R$	J_M	$-I_L$	$-K_L$	K_M	$-I_A$	J_L	J_A	J_R	$-K_A$	I_M	$-I_R$	$-K_Z$	E	$-I_Z$
K_Z	K_Z	$-J_L$	$-I_R$	K_M	$-J_A$	$-I_L$	I_M	K_R	K_L	K_A	$-J_R$	$-I_A$	J_M	$-J_Z$	$-I_Z$	E

THE 16×16 PRODUCT TABLE FOR THE POSITIVE “HEXPE” BASIS ELEMENTS

TABLE T.3 ⊢PAGE-I⊣

w_P IN TERMS OF THE COFACTOR F_{ij}:

$$
\begin{aligned}
w_0 &= (+F_{00} + F_{11} + F_{22} + F_{33})/4 \\
\\
w_{M1} &= (-F_{00} - F_{11} + F_{22} + F_{33})/4 \\
w_{M2} &= (-F_{00} + F_{11} - F_{22} + F_{33})/4 \\
w_{M3} &= (-F_{00} + F_{11} + F_{22} - F_{33})/4 \\
\\
w_{A1} &= (+F_{01} + F_{10} - F_{23} - F_{32})/4 \\
w_{A2} &= (+F_{02} - F_{13} + F_{20} - F_{31})/4 \\
w_{A3} &= (+F_{03} - F_{12} - F_{21} + F_{30})/4 \\
\\
w_{Z1} &= (-F_{01} - F_{10} - F_{23} - F_{32})/4 \\
w_{Z2} &= (-F_{02} - F_{13} - F_{20} - F_{31})/4 \\
w_{Z3} &= (-F_{03} - F_{12} - F_{21} - F_{30})/4 \\
\\
w_{R1} &= (+F_{01} - F_{10} + F_{23} - F_{32})/4 \\
w_{R2} &= (+F_{02} - F_{13} - F_{20} + F_{31})/4 \\
w_{R3} &= (+F_{03} + F_{12} - F_{21} - F_{30})/4 \\
\\
w_{L1} &= (+F_{01} - F_{10} - F_{23} + F_{32})/4 \\
w_{L2} &= (+F_{02} + F_{13} - F_{20} - F_{31})/4 \\
w_{L3} &= (+F_{03} - F_{12} + F_{21} - F_{30})/4
\end{aligned}
$$

w_P IN TERMS OF THE MINORS M_{ij}:

$$
\begin{aligned}
w_0 &= (+M_{00} + M_{11} + M_{22} + M_{33})/4 \\
\\
w_{M1} &= (-M_{00} - M_{11} + M_{22} + M_{33})/4 \\
w_{M2} &= (-M_{00} + M_{11} - M_{22} + M_{33})/4 \\
w_{M3} &= (-M_{00} + M_{11} + M_{22} - M_{33})/4 \\
\\
w_{A1} &= (-M_{01} - M_{10} + M_{23} + M_{32})/4 \\
w_{A2} &= (+M_{02} - M_{13} + M_{20} - M_{31})/4 \\
w_{A3} &= (-M_{03} + M_{12} + M_{21} - M_{30})/4 \\
\\
w_{Z1} &= (+M_{01} + M_{10} + M_{23} + M_{32})/4 \\
w_{Z2} &= (-M_{02} - M_{13} - M_{20} - M_{31})/4 \\
w_{Z3} &= (+M_{03} + M_{12} + M_{21} + M_{30})/4 \\
\\
w_{R1} &= (-M_{01} + M_{10} - M_{23} + M_{32})/4 \\
w_{R2} &= (+M_{02} - M_{13} - M_{20} + M_{31})/4 \\
w_{R3} &= (-M_{03} - M_{12} + M_{21} + M_{30})/4 \\
\\
w_{L1} &= (-M_{01} + M_{10} + M_{23} - M_{32})/4 \\
w_{L2} &= (+M_{02} + M_{13} - M_{20} - M_{31})/4 \\
w_{L3} &= (-M_{03} + M_{12} - M_{21} + M_{30})/4
\end{aligned}
$$

$$
\begin{aligned}
M_{00} &= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \\
M_{01} &= a_{10}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{20}a_{33} - a_{30}a_{23}) + a_{13}(a_{20}a_{32} - a_{30}a_{22}) \\
M_{02} &= a_{10}(a_{21}a_{33} - a_{31}a_{23}) - a_{11}(a_{20}a_{33} - a_{30}a_{23}) + a_{13}(a_{20}a_{31} - a_{30}a_{21}) \\
M_{03} &= a_{10}(a_{21}a_{32} - a_{31}a_{22}) - a_{11}(a_{20}a_{32} - a_{30}a_{22}) + a_{12}(a_{20}a_{31} - a_{30}a_{21}) \\
M_{10} &= a_{01}(a_{22}a_{33} - a_{32}a_{23}) - a_{02}(a_{21}a_{33} - a_{31}a_{23}) + a_{03}(a_{21}a_{32} - a_{31}a_{22}) \\
M_{11} &= a_{00}(a_{22}a_{33} - a_{32}a_{23}) - a_{02}(a_{20}a_{33} - a_{30}a_{23}) + a_{03}(a_{20}a_{32} - a_{30}a_{22}) \\
M_{12} &= a_{00}(a_{21}a_{33} - a_{31}a_{23}) - a_{01}(a_{20}a_{33} - a_{30}a_{23}) + a_{03}(a_{20}a_{31} - a_{30}a_{21}) \\
M_{13} &= a_{00}(a_{21}a_{32} - a_{31}a_{22}) - a_{01}(a_{20}a_{32} - a_{30}a_{22}) + a_{02}(a_{20}a_{31} - a_{30}a_{21}) \\
M_{20} &= a_{01}(a_{12}a_{33} - a_{32}a_{13}) - a_{02}(a_{11}a_{33} - a_{31}a_{13}) + a_{03}(a_{11}a_{32} - a_{31}a_{12}) \\
M_{21} &= a_{00}(a_{12}a_{33} - a_{32}a_{13}) - a_{02}(a_{10}a_{33} - a_{30}a_{13}) + a_{03}(a_{10}a_{32} - a_{30}a_{12}) \\
M_{22} &= a_{00}(a_{11}a_{33} - a_{31}a_{13}) - a_{01}(a_{10}a_{33} - a_{30}a_{13}) + a_{03}(a_{10}a_{31} - a_{30}a_{11}) \\
M_{23} &= a_{00}(a_{11}a_{32} - a_{31}a_{12}) - a_{01}(a_{10}a_{32} - a_{30}a_{12}) + a_{02}(a_{10}a_{31} - a_{30}a_{11}) \\
M_{30} &= a_{01}(a_{12}a_{23} - a_{22}a_{13}) - a_{02}(a_{11}a_{23} - a_{21}a_{13}) + a_{03}(a_{11}a_{22} - a_{21}a_{12}) \\
M_{31} &= a_{00}(a_{12}a_{23} - a_{22}a_{13}) - a_{02}(a_{10}a_{23} - a_{20}a_{13}) + a_{03}(a_{10}a_{22} - a_{20}a_{12}) \\
M_{32} &= a_{00}(a_{11}a_{23} - a_{21}a_{13}) - a_{01}(a_{10}a_{23} - a_{20}a_{13}) + a_{03}(a_{10}a_{21} - a_{20}a_{11}) \\
M_{33} &= a_{00}(a_{11}a_{22} - a_{21}a_{12}) - a_{01}(a_{10}a_{22} - a_{20}a_{12}) + a_{02}(a_{10}a_{21} - a_{20}a_{11})
\end{aligned}
$$

$w_0, \quad w_{M1}, \quad w_{M2}, \quad w_{M3}$

$$\begin{aligned}
a_{11}a_{22}a_{33} &= (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{11}a_{32}a_{23} &= (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{12}a_{21}a_{33} &= (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{12}a_{31}a_{23} &= (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{13}a_{21}a_{32} &= (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{13}a_{31}a_{22} &= (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{00}a_{22}a_{33} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{00}a_{32}a_{23} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{02}a_{20}a_{33} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{02}a_{30}a_{23} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{03}a_{20}a_{32} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{03}a_{30}a_{22} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{00}a_{11}a_{33} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{00}a_{31}a_{13} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \\
a_{01}a_{10}a_{33} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{01}a_{30}a_{13} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \\
a_{03}a_{10}a_{31} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \\
a_{03}a_{30}a_{11} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \\
a_{00}a_{11}a_{22} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{00}a_{21}a_{12} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \\
a_{01}a_{10}a_{22} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{01}a_{20}a_{12} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \\
a_{02}a_{10}a_{21} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \\
a_{02}a_{20}a_{11} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3})
\end{aligned}$$

$w_{R1}, \quad w_{L1}, \quad w_{A1}, \quad w_{Z1}$

$$\begin{aligned}
a_{10}a_{22}a_{33} &= (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{10}a_{32}a_{23} &= (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{12}a_{20}a_{33} &= (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{12}a_{30}a_{23} &= (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{13}a_{20}a_{32} &= (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{13}a_{30}a_{22} &= (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{01}a_{22}a_{33} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{01}a_{32}a_{23} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{02}a_{21}a_{33} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_0 + h_{M1} + h_{M2} - h_{M3}) \\
a_{02}a_{31}a_{23} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{03}a_{21}a_{32} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{03}a_{31}a_{22} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (+h_0 + h_{M1} - h_{M2} + h_{M3}) \\
a_{00}a_{11}a_{32} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{00}a_{31}a_{12} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \cdot (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \\
a_{01}a_{10}a_{32} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} - h_{L1} - h_{A1} - h_{Z1}) \\
a_{01}a_{30}a_{12} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (-h_{R3} + h_{L3} - h_{A3} - h_{Z3}) \\
a_{02}a_{10}a_{31} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (-h_{R2} + h_{L2} - h_{A2} - h_{Z2}) \\
a_{02}a_{30}a_{11} &= (-h_{R2} - h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R3} + h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \\
a_{00}a_{11}a_{23} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{00}a_{21}a_{13} &= (+h_0 - h_{M1} - h_{M2} - h_{M3}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \cdot (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \\
a_{01}a_{10}a_{23} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (-h_{R1} + h_{L1} - h_{A1} - h_{Z1}) \\
a_{01}a_{20}a_{13} &= (-h_{R1} - h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_{R2} - h_{L2} - h_{A2} - h_{Z2}) \\
a_{03}a_{10}a_{21} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R1} + h_{L1} + h_{A1} - h_{Z1}) \cdot (+h_{R3} - h_{L3} - h_{A3} - h_{Z3}) \\
a_{03}a_{20}a_{11} &= (-h_{R3} - h_{L3} + h_{A3} - h_{Z3}) \cdot (+h_{R2} + h_{L2} + h_{A2} - h_{Z2}) \cdot (+h_0 - h_{M1} + h_{M2} + h_{M3})
\end{aligned}$$

$w_{R2},\quad w_{L2},\quad w_{A2},\quad w_{Z2}$

$$
\begin{aligned}
a_{10}a_{21}a_{33} &= (+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{10}a_{31}a_{23} &= (+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{11}a_{20}a_{33} &= (+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{11}a_{30}a_{23} &= (+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{13}a_{20}a_{31} &= (+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\\
a_{13}a_{30}a_{21} &= (+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\\
a_{00}a_{21}a_{32} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{00}a_{31}a_{22} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{01}a_{20}a_{32} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{01}a_{30}a_{22} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{02}a_{20}a_{31} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\\
a_{02}a_{30}a_{21} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\\
a_{01}a_{12}a_{33} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{01}a_{32}a_{13} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{02}a_{11}a_{33} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{02}a_{31}a_{13} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{03}a_{11}a_{32} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{03}a_{31}a_{12} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\\
a_{00}a_{12}a_{23} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{00}a_{22}a_{13} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{02}a_{10}a_{23} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{02}a_{20}a_{13} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{03}a_{10}a_{22} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{03}a_{20}a_{12} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})
\end{aligned}
$$

$w_{R3},\quad w_{L3},\quad w_{A3},\quad w_{Z3}$

$$
\begin{aligned}
a_{10}a_{21}a_{32} &= (+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{10}a_{31}a_{22} &= (+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{11}a_{20}a_{32} &= (+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{11}a_{30}a_{22} &= (+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{12}a_{20}a_{31} &= (-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\\
a_{12}a_{30}a_{21} &= (-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\\
a_{00}a_{21}a_{33} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{00}a_{31}a_{23} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{01}a_{20}a_{33} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{01}a_{30}a_{23} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{03}a_{20}a_{31} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R2}+h_{L2}+h_{A2}-h_{Z2})\cdot(-h_{R2}+h_{L2}-h_{A2}-h_{Z2})\\
a_{03}a_{30}a_{21} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\\
a_{00}a_{12}a_{33} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{00}a_{32}a_{13} &= (+h_0-h_{M1}-h_{M2}-h_{M3})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{02}a_{10}a_{33} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(+h_0+h_{M1}+h_{M2}-h_{M3})\\
a_{02}a_{30}a_{13} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{03}a_{10}a_{32} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R1}+h_{L1}+h_{A1}-h_{Z1})\cdot(+h_{R1}-h_{L1}-h_{A1}-h_{Z1})\\
a_{03}a_{30}a_{12} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}+h_{L3}+h_{A3}-h_{Z3})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\\
a_{01}a_{12}a_{23} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{01}a_{22}a_{13} &= (-h_{R1}-h_{L1}+h_{A1}-h_{Z1})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{02}a_{11}a_{23} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(-h_{R1}+h_{L1}-h_{A1}-h_{Z1})\\
a_{02}a_{21}a_{13} &= (-h_{R2}-h_{L2}+h_{A2}-h_{Z2})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(+h_{R2}-h_{L2}-h_{A2}-h_{Z2})\\
a_{03}a_{11}a_{22} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_0-h_{M1}+h_{M2}+h_{M3})\cdot(+h_0+h_{M1}-h_{M2}+h_{M3})\\
a_{03}a_{21}a_{12} &= (-h_{R3}-h_{L3}+h_{A3}-h_{Z3})\cdot(+h_{R3}-h_{L3}-h_{A3}-h_{Z3})\cdot(-h_{R3}+h_{L3}-h_{A3}-h_{Z3})
\end{aligned}
$$

TABLE T.3 ⊢PAGE-IV⊣

HEXPE NUMBER

$$\begin{aligned} h = h_0\cdot\boldsymbol{E} &+ h_{M1}\cdot\boldsymbol{I}_M + h_{M2}\cdot\boldsymbol{J}_M + h_{M3}\cdot\boldsymbol{K}_M \\ + h_{R1}\cdot\boldsymbol{I}_R &+ h_{L1}\cdot\boldsymbol{I}_L + h_{A1}\cdot\boldsymbol{I}_A + h_{Z1}\cdot\boldsymbol{I}_Z \\ + h_{R2}\cdot\boldsymbol{J}_R &+ h_{L2}\cdot\boldsymbol{J}_L + h_{A2}\cdot\boldsymbol{J}_A + h_{Z2}\cdot\boldsymbol{I}_Z \\ + h_{R3}\cdot\boldsymbol{K}_R &+ h_{L3}\cdot\boldsymbol{K}_L + h_{A3}\cdot\boldsymbol{K}_A + h_{Z3}\cdot\boldsymbol{K}_Z \end{aligned}$$

HEXPE INVERSE

$$\begin{aligned} h^{-1} = (w_0\cdot\boldsymbol{E} &+ w_{M1}\cdot\boldsymbol{I}_M + w_{M2}\cdot\boldsymbol{J}_M + w_{M3}\cdot\boldsymbol{K}_M \\ + w_{R1}\cdot\boldsymbol{I}_R &+ w_{L1}\cdot\boldsymbol{I}_L + w_{A1}\cdot\boldsymbol{I}_A + w_{Z1}\cdot\boldsymbol{I}_Z \\ + w_{R2}\cdot\boldsymbol{J}_R &+ w_{L2}\cdot\boldsymbol{J}_L + w_{A2}\cdot\boldsymbol{J}_A + w_{Z2}\cdot\boldsymbol{I}_Z \\ + w_{R3}\cdot\boldsymbol{K}_R &+ w_{L3}\cdot\boldsymbol{K}_L + w_{A3}\cdot\boldsymbol{K}_A + w_{Z3}\cdot\boldsymbol{K}_Z)/d \end{aligned}$$

WEIGHT FACTORS

$w_0,\quad w_{M1},\quad w_{M2},\quad w_{M3}$

$$\begin{aligned} w_0 = &+h_0^3 + h_0(-h_{M1}^2 - h_{M2}^2 - h_{M3}^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2 - h_{A1}^2 - h_{A2}^2 - h_{A3}^2 - h_{Z1}^2 - h_{Z2}^2 - h_{Z3}^2) \\ &+ 2(-h_{M1}h_{M2}h_{M3} + h_{M1}h_{R1}h_{L1} + h_{M1}h_{A1}h_{Z1} + h_{M2}h_{R2}h_{L2} + h_{M2}h_{A2}h_{Z2} + h_{M3}h_{R3}h_{L3} + h_{M3}h_{A3}h_{Z3}) \\ &+ 2(+h_{R1}h_{L2}h_{A3} + h_{R1}h_{L3}h_{Z2} + h_{R2}h_{L1}h_{Z3} + h_{R2}h_{L3}h_{A1} + h_{R3}h_{L1}h_{A2} + h_{R3}h_{L2}h_{Z1} - h_{A1}h_{A2}h_{A3} - h_{Z1}h_{Z2}h_{Z3}) \end{aligned}$$

$$\begin{aligned} w_{M1} = &+h_{M1}^3 + h_{M1}(-h_0^2 - h_{M2}^2 - h_{M3}^2 + h_{R1}^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 - h_{L2}^2 - h_{L3}^2 - h_{A1}^2 + h_{A2}^2 + h_{A3}^2 - h_{Z1}^2 + h_{Z2}^2 + h_{Z3}^2) \\ &+ 2(-h_0h_{M2}h_{M3} + h_0h_{R1}h_{L1} + h_0h_{A1}h_{Z1} + h_{M2}h_{R3}h_{L3} + h_{M2}h_{A3}h_{Z3} + h_{M3}h_{R2}h_{L2} + h_{M3}h_{A2}h_{Z2}) \\ &+ 2(+h_{R1}h_{R2}h_{Z3} + h_{R1}h_{R3}h_{A2} - h_{R2}h_{L3}h_{Z1} - h_{R3}h_{L2}h_{A1} + h_{L1}h_{L2}h_{A3} + h_{L1}h_{L3}h_{Z2} + h_{A1}h_{Z2}h_{Z3} + h_{A2}h_{A3}h_{Z1}) \end{aligned}$$

$$\begin{aligned} w_{M2} = &+h_{M2}^3 + h_{M2}(-h_{M1}^2 - h_0^2 - h_{M3}^2 - h_{R1}^2 + h_{R2}^2 - h_{R3}^2 - h_{L1}^2 + h_{L2}^2 - h_{L3}^2 + h_{A1}^2 - h_{A2}^2 + h_{A3}^2 + h_{Z1}^2 - h_{Z2}^2 + h_{Z3}^2) \\ &+ 2(-h_0h_{M1}h_{M3} + h_0h_{R2}h_{L2} + h_0h_{A2}h_{Z2} + h_{M1}h_{R3}h_{L3} + h_{M1}h_{A3}h_{Z3} + h_{M3}h_{R1}h_{L1} + h_{M3}h_{A1}h_{Z1}) \\ &+ 2(+h_{R1}h_{R2}h_{A3} - h_{R1}h_{L3}h_{A2} + h_{R2}h_{R3}h_{Z1} - h_{R3}h_{L1}h_{Z2} + h_{L1}h_{L2}h_{Z3} + h_{L2}h_{L3}h_{A1} + h_{A1}h_{A3}h_{Z2} + h_{A2}h_{Z1}h_{Z3}) \end{aligned}$$

$$\begin{aligned} w_{M3} = &+h_{M3}^3 + h_{M3}(-h_{M1}^2 - h_{M2}^2 - h_0^2 - h_{R1}^2 - h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 + h_{L3}^2 + h_{A1}^2 + h_{A2}^2 - h_{A3}^2 + h_{Z1}^2 + h_{Z2}^2 - h_{Z3}^2) \\ &+ 2(-h_0h_{M1}h_{M2} + h_0h_{R3}h_{L3} + h_0h_{A3}h_{Z3} + h_{M1}h_{R2}h_{L2} + h_{M1}h_{A2}h_{Z2} + h_{M2}h_{R1}h_{L1} + h_{M2}h_{A1}h_{Z1}) \\ &+ 2(+h_{R1}h_{R3}h_{Z2} - h_{R1}h_{L2}h_{Z3} + h_{R2}h_{R3}h_{A1} - h_{R2}h_{L1}h_{A3} + h_{L1}h_{L3}h_{A2} + h_{L2}h_{L3}h_{Z1} + h_{A1}h_{A2}h_{Z3} + h_{A3}h_{Z1}h_{Z2}) \end{aligned}$$

$w_{R1},\quad w_{L1},\quad w_{A1},\quad w_{Z1}$

$$\begin{aligned} w_{R1} = &-h_{R1}^3 + h_{R1}(-h_{M1}^2 + h_{M2}^2 + h_{M3}^2 - h_0^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2 + h_{A1}^2 + h_{A2}^2 - h_{A3}^2 + h_{Z1}^2 - h_{Z2}^2 + h_{Z3}^2) \\ &+ 2(-h_0h_{M1}h_{L1} - h_0h_{L2}h_{A3} - h_0h_{L3}h_{Z2} - h_{M1}h_{R2}h_{Z3} - h_{M1}h_{R3}h_{A2} - h_{M2}h_{M3}h_{L1} - h_{M2}h_{R2}h_{A3}) \\ &+ 2(+h_{M2}h_{L3}h_{A2} - h_{M3}h_{R3}h_{Z2} + h_{M3}h_{L2}h_{Z3} - h_{R2}h_{A1}h_{Z2} - h_{R3}h_{A3}h_{Z1} + h_{L1}h_{A1}h_{Z1} - h_{L2}h_{A1}h_{A2} - h_{L3}h_{Z1}h_{Z3}) \end{aligned}$$

$$\begin{aligned} w_{L1} = &-h_{L1}^3 + h_{L1}(-h_{M1}^2 + h_{M2}^2 + h_{M3}^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_0^2 - h_{L2}^2 - h_{L3}^2 + h_{A1}^2 - h_{A2}^2 + h_{A3}^2 + h_{Z1}^2 + h_{Z2}^2 - h_{Z3}^2) \\ &+ 2(-h_0h_{M1}h_{R1} - h_0h_{R2}h_{Z3} - h_0h_{R3}h_{A2} - h_{M1}h_{L2}h_{A3} - h_{M1}h_{L3}h_{Z2} - h_{M2}h_{M3}h_{R1} + h_{M2}h_{R3}h_{Z2}) \\ &+ 2(-h_{M2}h_{L2}h_{Z3} + h_{M3}h_{R2}h_{A3} - h_{M3}h_{L3}h_{A2} + h_{R1}h_{A1}h_{Z1} - h_{R2}h_{Z1}h_{Z2} - h_{R3}h_{A1}h_{A3} - h_{L2}h_{A2}h_{Z1} - h_{L3}h_{A1}h_{Z3}) \end{aligned}$$

$$\begin{aligned} w_{A1} = &+h_{A1}^3 + h_{A1}(-h_{M1}^2 + h_{M2}^2 + h_{M3}^2 - h_{R1}^2 + h_{R2}^2 - h_{R3}^2 - h_{L1}^2 - h_{L2}^2 + h_{L3}^2 - h_0^2 - h_{A2}^2 - h_{A3}^2 - h_{Z1}^2 + h_{Z2}^2 + h_{Z3}^2) \\ &+ 2(+h_0h_{M1}h_{Z1} + h_0h_{R2}h_{L3} - h_0h_{A2}h_{A3} - h_{M1}h_{R3}h_{L2} + h_{M1}h_{Z2}h_{Z3} + h_{M2}h_{M3}h_{Z1} + h_{M2}h_{L2}h_{L3}) \\ &+ 2(+h_{M2}h_{A3}h_{Z2} + h_{M3}h_{R2}h_{R3} + h_{M3}h_{A2}h_{Z3} + h_{R1}h_{R2}h_{Z2} - h_{R1}h_{L1}h_{Z1} + h_{R1}h_{L2}h_{A2} + h_{R3}h_{L1}h_{A3} + h_{L1}h_{L3}h_{Z3}) \end{aligned}$$

$$\begin{aligned} w_{Z1} = &+h_{Z1}^3 + h_{Z1}(-h_{M1}^2 + h_{M2}^2 + h_{M3}^2 - h_{R1}^2 - h_{R2}^2 + h_{R3}^2 - h_{L1}^2 + h_{L2}^2 - h_{L3}^2 - h_{A1}^2 + h_{A2}^2 + h_{A3}^2 - h_0^2 - h_{Z2}^2 - h_{Z3}^2) \\ &+ 2(+h_0h_{M1}h_{A1} + h_0h_{R3}h_{L2} - h_0h_{Z2}h_{Z3} - h_{M1}h_{R2}h_{L3} + h_{M1}h_{A2}h_{A3} + h_{M2}h_{M3}h_{A1} + h_{M2}h_{R2}h_{R3}) \\ &+ 2(+h_{M2}h_{A2}h_{Z3} + h_{M3}h_{L2}h_{L3} + h_{M3}h_{A3}h_{Z2} + h_{R1}h_{R3}h_{A3} - h_{R1}h_{L1}h_{A1} + h_{R1}h_{L3}h_{Z3} + h_{R2}h_{L1}h_{Z2} + h_{L1}h_{L2}h_{A2}) \end{aligned}$$

TABLE T.3 $\vdash$PAGE-V$\dashv$

$$w_{R2},\quad w_{L2},\quad w_{A2},\quad w_{Z2}$$

$$\begin{aligned}w_{R2} = &-h_{R2}^3 + h_{R2}(+h_{M1}^2 - h_{M2}^2 + h_{M3}^2 - h_{R1}^2 - h_0^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2 - h_{A1}^2 + h_{A2}^2 + h_{A3}^2 + h_{Z1}^2 + h_{Z2}^2 - h_{Z3}^2)\\ &+ 2(-h_0h_{M2}h_{L2} - h_0h_{L1}h_{Z3} - h_0h_{L3}h_{A1} - h_{M1}h_{M3}h_{L2} - h_{M1}h_{R1}h_{Z3} + h_{M1}h_{L3}h_{Z1} - h_{M2}h_{R1}h_{A3})\\ &+ 2(-h_{M2}h_{R3}h_{Z1} - h_{M3}h_{R3}h_{A1} + h_{M3}h_{L1}h_{A3} - h_{R1}h_{A1}h_{Z2} - h_{R3}h_{A2}h_{Z3} - h_{L1}h_{Z1}h_{Z2} + h_{L2}h_{A2}h_{Z2} - h_{L3}h_{A2}h_{A3})\end{aligned}$$

$$\begin{aligned}w_{L2} = &-h_{L2}^3 + h_{L2}(+h_{M1}^2 - h_{M2}^2 + h_{M3}^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_0^2 - h_{L3}^2 + h_{A1}^2 + h_{A2}^2 - h_{A3}^2 - h_{Z1}^2 + h_{Z2}^2 + h_{Z3}^2)\\ &+ 2(-h_0h_{M2}h_{R2} - h_0h_{R1}h_{A3} - h_0h_{R3}h_{Z1} - h_{M1}h_{M3}h_{R2} + h_{M1}h_{R3}h_{A1} - h_{M1}h_{L1}h_{A3} - h_{M2}h_{L1}h_{Z3})\\ &+ 2(-h_{M2}h_{L3}h_{A1} + h_{M3}h_{R1}h_{Z3} - h_{M3}h_{L3}h_{Z1} - h_{R1}h_{A1}h_{A2} + h_{R2}h_{A2}h_{Z2} - h_{R3}h_{Z2}h_{Z3} - h_{L1}h_{A2}h_{Z1} - h_{L3}h_{A3}h_{Z2})\end{aligned}$$

$$\begin{aligned}w_{A2} = &+h_{A2}^3 + h_{A2}(+h_{M1}^2 - h_{M2}^2 + h_{M3}^2 - h_{R1}^2 - h_{R2}^2 + h_{R3}^2 + h_{L1}^2 - h_{L2}^2 - h_{L3}^2 - h_{A1}^2 - h_0^2 - h_{A3}^2 + h_{Z1}^2 - h_{Z2}^2 + h_{Z3}^2)\\ &+ 2(+h_0h_{M2}h_{Z2} + h_0h_{R3}h_{L1} - h_0h_{A1}h_{A3} + h_{M1}h_{M3}h_{Z2} + h_{M1}h_{R1}h_{R3} + h_{M1}h_{A3}h_{Z1} - h_{M2}h_{R1}h_{L3})\\ &+ 2(+h_{M2}h_{Z1}h_{Z3} + h_{M3}h_{L1}h_{L3} + h_{M3}h_{A1}h_{Z3} + h_{R1}h_{L2}h_{A1} + h_{R2}h_{R3}h_{Z3} - h_{R2}h_{L2}h_{Z2} + h_{R2}h_{L3}h_{A3} + h_{L1}h_{L2}h_{Z1})\end{aligned}$$

$$\begin{aligned}w_{Z2} = &+h_{Z2}^3 + h_{Z2}(+h_{M1}^2 - h_{M2}^2 + h_{M3}^2 + h_{R1}^2 - h_{R2}^2 - h_{R3}^2 - h_{L1}^2 - h_{L2}^2 + h_{L3}^2 + h_{A1}^2 - h_{A2}^2 + h_{A3}^2 - h_{Z1}^2 - h_0^2 - h_{Z3}^2)\\ &+ 2(+h_0h_{M2}h_{A2} + h_0h_{R1}h_{L3} - h_0h_{Z1}h_{Z3} + h_{M1}h_{M3}h_{A2} + h_{M1}h_{L1}h_{L3} + h_{M1}h_{A1}h_{Z3} - h_{M2}h_{R3}h_{L1})\\ &+ 2(+h_{M2}h_{A1}h_{A3} + h_{M3}h_{R1}h_{R3} + h_{M3}h_{A3}h_{Z1} + h_{R1}h_{R2}h_{A1} + h_{R2}h_{L1}h_{Z1} - h_{R2}h_{L2}h_{A2} + h_{R3}h_{L2}h_{Z3} + h_{L2}h_{L3}h_{A3})\end{aligned}$$

$$w_{R3},\quad w_{L3},\quad w_{A3},\quad w_{Z3}$$

$$\begin{aligned}w_{R3} = &-h_{R3}^3 + h_{R3}(+h_{M1}^2 + h_{M2}^2 - h_{M3}^2 - h_{R1}^2 - h_{R2}^2 - h_0^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2 + h_{A1}^2 - h_{A2}^2 + h_{A3}^2 - h_{Z1}^2 + h_{Z2}^2 + h_{Z3}^2)\\ &+ 2(-h_0h_{M3}h_{L3} - h_0h_{L1}h_{A2} - h_0h_{L2}h_{Z1} - h_{M1}h_{M2}h_{L3} - h_{M1}h_{R1}h_{A2} + h_{M1}h_{L2}h_{A1} - h_{M2}h_{R2}h_{Z1})\\ &+ 2(+h_{M2}h_{L1}h_{Z2} - h_{M3}h_{R1}h_{Z2} - h_{M3}h_{R2}h_{A1} - h_{R1}h_{A3}h_{Z1} - h_{R2}h_{A2}h_{Z3} - h_{L1}h_{A1}h_{A3} - h_{L2}h_{Z2}h_{Z3} + h_{L3}h_{A3}h_{Z3})\end{aligned}$$

$$\begin{aligned}w_{L3} = &-h_{L3}^3 + h_{L3}(+h_{M1}^2 + h_{M2}^2 - h_{M3}^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 - h_0^2 - h_{A1}^2 + h_{A2}^2 + h_{A3}^2 + h_{Z1}^2 - h_{Z2}^2 + h_{Z3}^2)\\ &+ 2(-h_0h_{M3}h_{R3} - h_0h_{R1}h_{Z2} - h_0h_{R2}h_{A1} - h_{M1}h_{M2}h_{R3} + h_{M1}h_{R2}h_{Z1} - h_{M1}h_{L1}h_{Z2} + h_{M2}h_{R1}h_{A2})\\ &+ 2(-h_{M2}h_{L2}h_{A1} - h_{M3}h_{L1}h_{A2} - h_{M3}h_{L2}h_{Z1} - h_{R1}h_{Z1}h_{Z3} - h_{R2}h_{A2}h_{A3} + h_{R3}h_{A3}h_{Z3} - h_{L1}h_{A1}h_{Z3} - h_{L2}h_{A3}h_{Z2})\end{aligned}$$

$$\begin{aligned}w_{A3} = &+h_{A3}^3 + h_{A3}(+h_{M1}^2 + h_{M2}^2 - h_{M3}^2 + h_{R1}^2 - h_{R2}^2 - h_{R3}^2 - h_{L1}^2 + h_{L2}^2 - h_{L3}^2 - h_{A1}^2 - h_{A2}^2 - h_0^2 + h_{Z1}^2 + h_{Z2}^2 - h_{Z3}^2)\\ &+ 2(+h_0h_{M3}h_{Z3} + h_0h_{R1}h_{L2} - h_0h_{A1}h_{A2} + h_{M1}h_{M2}h_{Z3} + h_{M1}h_{L1}h_{L2} + h_{M1}h_{A2}h_{Z1} + h_{M2}h_{R1}h_{R2})\\ &+ 2(+h_{M2}h_{A1}h_{Z2} - h_{M3}h_{R2}h_{L1} + h_{M3}h_{Z1}h_{Z2} + h_{R1}h_{R3}h_{Z1} + h_{R2}h_{L3}h_{A2} + h_{R3}h_{L1}h_{A1} - h_{R3}h_{L3}h_{Z3} + h_{L2}h_{L3}h_{Z2})\end{aligned}$$

$$\begin{aligned}w_{Z3} = &+h_{Z3}^3 + h_{Z3}(+h_{M1}^2 + h_{M2}^2 - h_{M3}^2 - h_{R1}^2 + h_{R2}^2 - h_{R3}^2 + h_{L1}^2 - h_{L2}^2 - h_{L3}^2 + h_{A1}^2 + h_{A2}^2 - h_{A3}^2 - h_{Z1}^2 - h_{Z2}^2 - h_0^2)\\ &+ 2(+h_0h_{M3}h_{A3} + h_0h_{R2}h_{L1} - h_0h_{Z1}h_{Z2} + h_{M1}h_{M2}h_{A3} + h_{M1}h_{R1}h_{R2} + h_{M1}h_{A1}h_{Z2} + h_{M2}h_{L1}h_{L2})\\ &+ 2(+h_{M2}h_{A2}h_{Z1} - h_{M3}h_{R1}h_{L2} + h_{M3}h_{A1}h_{A2} + h_{R1}h_{L3}h_{Z1} + h_{R2}h_{R3}h_{A2} + h_{R3}h_{L2}h_{Z2} - h_{R3}h_{L3}h_{A3} + h_{L1}h_{L3}h_{A1})\end{aligned}$$

DETERMINANT

$$d = \begin{vmatrix} a_{00} & a_{01} & a_{02} & a_{03}\\ a_{10} & a_{11} & a_{12} & a_{13}\\ a_{20} & a_{21} & a_{22} & a_{23}\\ a_{30} & a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{00}M_{00} - a_{01}M_{01} + a_{02}M_{02} - a_{03}M_{03}$$

$$\begin{aligned}&= h_0w_0 + h_{M1}w_{M1} + h_{M2}w_{M2} + h_{M3}w_{M3} + h_{A1}w_{A1} + h_{A2}w_{A2} + h_{A3}w_{A3} + h_{Z1}w_{Z1} + h_{Z2}w_{Z2} + h_{Z3}w_{Z3}\\ &- h_{R1}w_{R1} - h_{R2}w_{R2} - h_{R3}w_{R3} - h_{L1}w_{L1} - h_{L2}w_{L2} - h_{L3}w_{L3}\end{aligned}$$

TABLE T.4 ⊢PAGE-A⊣

OCTIVOS PRODUCT TABLES

·	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
1	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
i	i	-1	k	-j	I	-E	-K	J	-i	1	-k	j	-I	E	K	-J
j	j	-k	-1	i	J	K	-E	-I	-j	k	1	-i	-J	-K	E	I
k	k	j	-i	-1	K	-J	I	-E	-k	-j	i	1	-K	J	-I	E
E	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k	E	I	J	K
I	-i	1	-k	j	-I	E	K	-J	i	-1	k	-j	I	-E	-K	J
J	-j	k	1	-i	-J	-K	E	I	j	-k	-1	i	J	K	-E	-I
K	-k	-j	i	1	-K	J	-I	E	k	j	-i	-1	K	-J	I	-E
-1	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k	E	I	J	K
-i	-i	1	-k	j	-I	E	K	-J	i	-1	k	-j	I	-E	-K	J
-j	-j	k	1	-i	-J	-K	E	I	j	-k	-1	i	J	K	-E	-I
-k	-k	-j	i	1	-K	J	-I	E	k	j	-i	-1	K	-J	I	-E
-E	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
-I	i	-1	k	-j	I	-E	-K	J	-i	1	-k	j	-I	E	K	-J
-J	j	-k	-1	i	J	K	-E	-I	-j	k	1	-i	-J	-K	E	I
-K	k	j	-i	-1	K	-J	I	-E	-k	-j	i	1	-K	J	-I	E
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

SPLIT OCTIVOS $\longrightarrow (A,B)(C,D) = (AC - B^H C, A^H D - BD)$
$A, C \in \mathbb{H}_R,\ \ B, D \in \mathbb{H}_L$

Let Hamilton's quaternion R-H basis be, $\{1, i, j, k\}$, and L-H basis be, $\{1, I, J, K\}$, so that, $ij = +k, ...,\ IJ = -K, ...,$ etc.. then re-define the letters by the ordered pairs,

$1 = (1,0), i = (i,0), j = (j,0), k = (k,0)$
$E = (0,1), I = (0,I), J = (0,J), K = (0,K)$
etc.. then, for example, since, $i^H = I$,
$iJ = (i,0)(0,J) = (0,IJ) = (0,-K) = -K$

Then, for the Split Octivos, the R-H, and L-H form separate algebras.
$(A,0)(C,0) = +(AC,0)$ R-H×R-H=R-H
$(0,B)(0,D) = -(0,BD)$ L-H×L-H=L-H
while also providing an interpretation for a product of right hand with left hand.
$(A,0)(0,D) = (0,A^H D)$ R-H×L-H=L-H
$(0,B)(C,0) = (-B^H C,0)$ L-H×R-H=R-H
Let $A_R = (A,0)$ and $D_L = p(0,D)$, where $p^2 = -1$, and p commutes with all numbers like a scalar. Then, note that,

$$A_R D_L \neq D_L A_R,\ \ A_R \in \mathbb{H}_R,\ D_L \in \breve{\mathbb{H}}_L$$

Right and left quaternions do not commute here, the way they do in the `hexpe algebra`. Also, our L-H is a little different from usual, $\mathbb{H}_L$, since, $\breve{\mathbb{H}}_L$, doesn't share scalars with the R-H algebra, $\mathbb{H}_R$. They are truly split.

·	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
1	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
i	i	-1	k	-j	I	-E	-K	J	-i	1	-k	j	-I	E	K	-J
j	j	-k	-1	i	J	K	-E	-I	-j	k	1	-i	-J	-K	E	I
k	k	j	-i	-1	K	-J	I	-E	-k	-j	i	1	-K	J	-I	E
E	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k
I	I	-E	K	-J	-i	1	k	-j	-I	E	-K	J	i	-1	-k	j
J	J	-K	-E	I	-j	-k	1	i	-J	K	E	-I	j	k	-1	-i
K	K	J	-I	-E	-k	j	-i	1	-K	-J	I	E	k	-j	i	-1
-1	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k	E	I	J	K
-i	-i	1	-k	j	-I	E	K	-J	i	-1	k	-j	I	-E	-K	J
-j	-j	k	1	-i	-J	-K	E	I	j	-k	-1	i	J	K	-E	-I
-k	-k	-j	i	1	-K	J	-I	E	k	j	-i	-1	K	-J	I	-E
-E	-E	-I	-J	-K	1	i	j	k	E	I	J	K	-1	-i	-j	-k
-I	-I	E	-K	J	i	-1	-k	j	I	-E	K	-J	-i	1	k	-j
-J	-J	K	E	-I	j	k	-1	-i	J	-K	-E	I	-j	-k	1	i
-K	-K	-J	I	E	k	-j	i	-1	K	J	-I	-E	-k	j	-i	1
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

There are two left identity elements, $\{1, -E\}$, and no right identities. Every column has a 1 or $-E$, but not both. So, every element has an inverse to 1 or $-E$.

The Plain Octivos have a single identity, which is both a left and a right identity, but the R-H and L-H are now fully integrated into the Octivo algebra and can no longer be separated. The ordered pair, $(A,0)$, can be identified with, $\mathbb{H}_R$. But, $(0,D)$ is now so different from D that it loses its special L-H character. It is no longer simply proportional to an element of $\mathbb{H}_L$. $(0,B)\cdot(0,D) \in \mathbb{H}_R$!

PLAIN OCTIVOS $\longrightarrow (A,B)(C,D) = (AC - (BD)^H, A^H D + C^H B)$
$A, C \in \mathbb{H}_R,\ \ B, D \in \mathbb{H}_L$

However, the squares of the basis elements now have the values, +1 or −1, instead of the four values found in Split Octivos.

The Plain Octivos are closer in spirit to the Cayley-Dickson construction than the Split Octivos.

TABLE T.4 ⊢PAGE-B⊣

OCTIVOS . PRODUCT TABLES

·	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
1	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
i	i	-1	k	-j	-I	E	K	-J	-i	1	-k	j	I	-E	-K	J
j	j	-k	-1	i	-J	-K	E	I	-j	k	1	-i	J	K	-E	-I
k	k	j	-i	-1	-K	J	-I	E	-k	-j	i	1	K	-J	I	-E
E	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k
I	I	-E	K	-J	i	-1	-k	j	-I	E	-K	J	-i	1	k	-j
J	J	-K	-E	I	j	k	-1	-i	-J	K	E	-I	-j	-k	1	i
K	K	J	-I	-E	k	-j	i	-1	-K	-J	I	E	-k	j	-i	1
-1	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k	E	I	J	K
-i	-i	1	-k	j	I	-E	-K	J	i	-1	k	-j	-I	E	K	-J
-j	-j	k	1	-i	J	K	-E	-I	j	-k	-1	i	-J	-K	E	I
-k	-k	-j	i	1	K	-J	I	-E	k	j	-i	-1	-K	J	-I	E
-E	-E	-I	-J	-K	1	i	j	k	E	I	J	K	-1	-i	-j	-k
-I	-I	E	-K	J	-i	1	k	-j	I	-E	K	-J	i	-1	-k	j
-J	-J	K	E	-I	-j	-k	1	i	J	-K	-E	I	j	k	-1	-i
-K	-K	-J	I	E	-k	j	-i	1	K	J	-I	-E	k	-j	i	-1
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

CONJUGATED OCTIVOS $\longrightarrow (A,B)(C,D) = (AC - (B^*D)^H, (A^*)^H D + C^H B)$
$A, C \in \mathbb{H}_R,\ \ B, D \in \mathbb{H}_L$

·	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
1	1	i	j	k	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K
i	i	-1	k	-j	-I	E	-K	J	-i	1	-k	j	I	-E	K	-J
j	j	-k	-1	i	-J	K	E	-I	-j	k	1	-i	J	-K	-E	I
k	k	j	-i	-1	-K	-J	I	E	-k	-j	i	1	K	J	-I	-E
E	E	I	J	K	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k
I	I	-E	-K	J	i	-1	-k	j	-I	E	K	-J	-i	1	k	-j
J	J	K	-E	-I	j	k	-1	-i	-J	-K	E	I	-j	-k	1	i
K	K	-J	I	-E	k	-j	i	-1	-K	J	-I	E	-k	j	-i	1
-1	-1	-i	-j	-k	-E	-I	-J	-K	1	i	j	k	E	I	J	K
-i	-i	1	-k	j	I	-E	K	-J	i	-1	k	-j	-I	E	-K	J
-j	-j	k	1	-i	J	-K	-E	I	j	-k	-1	i	-J	K	E	-I
-k	-k	-j	i	1	K	J	-I	-E	k	j	-i	-1	-K	-J	I	E
-E	-E	-I	-J	-K	1	i	j	k	E	I	J	K	-1	-i	-j	-k
-I	-I	E	K	-J	-i	1	k	-j	I	-E	-K	J	i	-1	-k	j
-J	-J	-K	E	I	-j	-k	1	i	J	K	-E	-I	j	k	-1	-i
-K	-K	J	-I	E	-k	j	-i	1	K	-J	I	-E	k	-j	i	-1
	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.

OCTONIONS (OCTAVES) $\longrightarrow (A,B)(C,D) = (AC - DB^*, A^*D + CB)$
$A, B, C, D \in \mathbb{H}_R$
=or=
ISOMORPH OCTIVOS $\longrightarrow (A,B)(C,D) = (AC - (D^*B)^H, D^*A^H + C^H B)$
$A, C \in \mathbb{H}_R,\ \ B, D \in \mathbb{H}_L$

The Conjugated Octivos make use of both the hand transformation operator and conjugation in a Cayley-Dickson type construction.

Here the squares of all the imaginary elements are -1, like complex numbers, quaternions, and octonions. The unique identity exists, and every element has an unique inverse. Once again, the L-H is fully integrated with the R-H in this Octivo system, and $(0, D)$ can't be treated like a proportional member of $\mathbb{H}_L$, even though $D \in \mathbb{H}_L$.

For the Octonions, we use only Hamilton R-H basis $ij = +k$, ..., etc.. and define the labels, $1 = (1, 0), i = (i, 0), j = (j, 0), k = (k, 0)$, $E = (0, 1), I = (0, i), J = (0, j), K = (0, k)$. Here the ordered pairs are built from two R-H quaternion systems.

Then, with a small modification to the conjugated octivos we obtain a Cayley-Dickson type construction that creates an algebra with the same product table as the octonions, yet are constructed with a pair of R-H and L-H quaternions, instead of the usual two right hands. We call this new algebra "`isomorph octivos`".

Given that Hexadecanions (sedenions) are constructed from a pair of octonions, and we now have isomorph octivos that are essentially equivalent in form, we can construct another 16-dimensional algebra isomorphic to the hexadecanions, using two R-H and two L-H quaternions instead.

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[1] W. R. Hamilton, 1844, *On a new species of Imaginary Quantities connected with the Theory of Quaternions* [communicated November 13, 1843], Ir. Acad. Proc., II, 424-434.

[2] In his "*Lectures on Quaternions*" Dublin 1853, pp.61-65, Hamilton describes the movements of a telescope using his new `ijk` elements. Here south is $+i$, west is $+j$, and upwards is $+k$, so that the "right-hand rule" $ij = +k$, represents the action of south turning towards west producing an upward result, whereas today in the modern definition of the right-hand rule, south turning towards west produces a downward result. `see book online at: http://historical.library.cornell.edu/`

[3] We may replace the elemental right-hand basis $\{1, i, j, k\}$ by any representation, like the matrices $\{\mathbf{1}, \boldsymbol{I}, \boldsymbol{J}, \boldsymbol{K}\}$

[4] We shall often use $\boldsymbol{E}$ for the unit matrix, when it makes things clearer than the sometimes confusing symbol $\mathbf{1}$. However, we won't necessarily give all the product rules for $\boldsymbol{E}$ in our definitions—it being understood that $\boldsymbol{E}$, like $\mathbf{1}$, commutes with every other number, and leaves that number unchanged, i.e. $\boldsymbol{EZ} = \boldsymbol{ZE} = \boldsymbol{Z}$, regardless of what $\boldsymbol{Z}$ is given.

[5] That is to say, when we fix `R` before `L`, then we have cyclic, $I_A = J_R K_L$, and acyclic, $I_Z = K_R J_L$. But, if we fix `L` before `R`, we have the reverse situation, with acyclic, $I_A = K_L J_R$, and now cyclic, $I_Z = J_L K_R$. Right hand elements commute with left hand elements, so we must pick a convention to describe the relative behaviors of the `A` and `Z` numbers—we chose to put `R` before `L`.

[6] Davenport, C. M., *A Commutative Hypercomplex Calculus with Applications to Special Relativity* (Privately published, Knoxville, Tennessee, 1991)

[7] Davenport, C. M., "A Commutative Hypercomplex Algebra with Associated Function Theory." In *Clifford Algebras with Numeric and Symbolic Computations*, R. Ablamowicz, Ed. Birkhauser Bosten, 1996, pp. 213–227.

[8] The `R-H` and `L-H` quaternion algebras are distinct by virtue of the handedness property $ijk = \pm 1$, but are otherwise isomorphic to each other. While, the `M-H`, `A-H`, `Z-H` algebras have no such distinguishing characteristic to differentiate them at all, and are therefore more identical.

[9] We either consider the five–`R,L,M,A,Z`—4-d numbers, or alternatively consider the algebras from the `five` groups of order eight, to suggest this naming convention.

[10] Well, HEXADECAPENTAQUATERNIONS was just a little too long, so we had to shorten it to HEXPENTAQUATERNIONS, then to HEXPE.

[11] Hamilton called this term, $h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2$, the '`norm`' of the quaternion. But sometimes mathematicians define the '`norm`' to be the `square-root` of this same expression. So, we shall use the term '`square norm`' to emphasize we're talking about the sum-of-squares when referring to this type of factor.

[12] $h_M'^{\dagger} = (w_0 \boldsymbol{E} + w_1 \boldsymbol{I}_M + w_2 \boldsymbol{J}_M + w_3 \boldsymbol{K}_M) \,/\, \left[N_M^4\right]^{3/4}$

[13] W. M. Petrie, "*The Pyramids and Temples of Gizeh*" (1883)—online: (see Ch 6. Sec.24. for base angles) `http://www.ronaldbirdsall.com/gizeh/`

[14] It may also be interesting to note that, $5 \times 73 = 365$, and that, $4 \times 73 = 292$, which is the 5-th element of the continued fraction in PI $= \{3; 7, 15, 1, 292, 1, ...\}$.

[15] It is to be understood here that, $Z' \cdot Y = Y \cdot Z$, expresses the condition that there is a fixed Z', counterpart to a given Z, that makes this equation true for all Y in the algebra under consideration.

[16] Let the square's vertices be labeled 1-2-3-4 in clockwise order, and let R90, R180, R270, all represent clockwise rotations. The lines of reflection, D13, D24, B12, B23, are fixed in the plane of the square to the initial square's placement and don't move around with the vertices of the square. Then, the left column element of the product table, e.g. R90, represents the first operation, and the top row the second element, e.g. D13, in the binary product, e.g. $R90 \cdot D13 = B23$.

[17] This is the Schönflies notation convention used mostly by biochemists and spectroscopists. Mathematicians often use $\mathbb{Z}_n$ for the cyclic group, rather than C_n, and with the exception of the symmetric group symbol, S_n, used in mathematical literature, most of the biochemist symbols are also adopted by mathematicians. The alternative crystallographic Hermann-Mauguin nomenclature used to identify groups is also found in math papers.

[18] This also causes parallel lines to converge. A projection that maps 3-d objects onto the plane while preserving parallel lines is an "`orthographic projection`."

[19] W. R. Hamilton, June 1845, BAAS.

[20] The space inversion described by the Parity operator, $P\colon (x, y, z) \mapsto (-x, -y, -z)$, is equivalent to a reflection in the yz-plane, $I_X\colon (x, y, z) \mapsto (-x, y, z)$, followed by a $180°$ rotation about the x-axis normal to that plane, $R_X(180)\colon (x, y, z) \mapsto (x, -y, -z)$, i.e.
$P \equiv I_X R_X(180) \equiv I_Y R_Y(180) \equiv I_Z R_Z(180)$.
So, Parity combines a plane mirror with a half turn.

[21] Lord Kelvin (William Thomson), Baltimore Lectures on Molecular Dynamics and The Wave Theory of Light, 1884.

[22] McAulay, Alexander, *Utility of Quaternions in Physics*, London: Macmillan and Co., xiv + 107pp, 1893.;

[23] P. M. Jack, "Physical Space as a Quaternion Structure: Maxwell's Equations. A Brief Note", `http://www.arxiv.org/abs/math-ph/0307038`

[24] Given particular boundary conditions, we can then use integral transforms to convert this equation to an ordinary matrix equation, solve with matrix algebra or hexpe algebra, then take the inverse integral transform to get the result.

[25] Note 5 elements $\{e, i_R, j_R, i_L, j_L\}$ generate this algebra, since, k_R, k_L, can also be considered "defined" by these, and hence rules involving k's are also "derived" rules.

[26] Summation over the repeated indices, p, q, r, s, is implied. Every hexpe number, h, can be written as the sum of RL pairs , $h = P_{11p} Q'_{p11} = P_{11,1} Q'_{1,11} + P_{11,2} Q'_{2,11} + \ldots + P_{11,m} Q'_{m,11}$, where the Ps are R-H, and Q's L-H.

[27] Hamilton describes his struggles searching for triplets in "*Lectures on Quaternions*", p.16, he first starts looking for triplets in 1833, after he'd invented "doublets", which are ordered pairs (x, y) equivalent to complex numbers $x + iy$, and first publically mentions his triplet attempts at the end of an 1835 Essay. It then takes him until 1843 before he obtains his remarkable insight that the 4th parameter is essential.

[28] Also called SEDENIONS.

[29] An alternate Cayley-Dickson construction definition is,

$(A, B)(C, D) = (AC - D^*B, BC^* + DA)$. There's also a more modern "Conway-Smith process" defined by,
$(A, B)(C, D) = (AC - (B^*D)^*, BC^* + B(A^*(B^{-1}D)^*)^*)$
$(A, B)(C, D) = (AC - (B^*D)^*, BC^* + (A^*D^*)^*)$
1st definition used when $B \neq 0$, and the 2nd if $B = 0$.

[30] In fact all associative algebras can be represented by matrix algebra. However, the `hexpe algebra` is equivalent to the complete matrix algebra of 4×4 matrices, it is not a subalgebra of the matrix algebra. It is just a particular hypercomplex decomposition of that matrix algebra.

[31] Recall that left hand quaternions are "`isomorphic`" to right hand quaternions. This question can then be generalized—is there a way to use isomorphic lower dimensional algebras in the construction of the higher dimensional one, with a Cayley-Dickson type construction, instead of requiring the lower algebras be identical?

[32] The names "`octonions`" and "`octaves`" being already taken, we've settled on "`octivos`" to call the whole class of 8-d numbers that can be formed from a doublet product using a pair of "right and left" quaternions; this includes other constructions, not mentioned here in this paper. The versions mentioned here may then be referred to by the terms: (1) "`split octivos`," since (A,0) and (0,B) form separate R-H and L-H quaternion algebras, but there are effectively two identity elements; (2) "`plain octivos`," since only the hand operator is used in the formula; (3) "`conjugated octivos`," since both the conjugation and hand transformation are used in construction; (4) "`isomorph octivos`," since this construction is isomorphic to the octonions.

Quatro-Quaternions
and the matrix representations of octonions.

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We introduce a modified product that enables matrix representations of octonions and all other Cayley-Dickson algebras, and discuss a new dual-product matrix algebra called quatro-quaternions.

I. INTRODUCTION.

The reals, complex numbers, and quaternions, can be represented by matrix algebra. But, since the standard matrix algebra has an associative multiplication it cannot normally represent the octonions. However, with a simple modification to the definition of the matrix product one can also represent octonions. Matrices can be extended to include both an "associative product" and a "non-associative product," resulting in a more flexible matrix algebra capable of representing all the normed division algebras, and indeed all Cayley-Dickson algebras as well. Consider the following two alternative expressions for 2×2 matrix multiplication.

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \cdot \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} A_{00}B_{00} + A_{01}B_{10} & A_{00}B_{01} + A_{01}B_{11} \\ A_{10}B_{00} + A_{11}B_{10} & A_{10}B_{01} + A_{11}B_{11} \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \times \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} A_{00}B_{00} + B_{10}A_{01} & B_{01}A_{00} + A_{01}B_{11} \\ B_{00}A_{10} + A_{11}B_{10} & A_{10}B_{01} + B_{11}A_{11} \end{pmatrix} \tag{2}$$

When the multiplication of the matrix entries commute, $A_{01}B_{10} = B_{10}A_{01}$ etc., there is no difference between these two expressions. But, when these entries non-commute, $A_{01}B_{10} \neq B_{10}A_{01}$, the results of the two formulas are generally different. When the A's and B's are quaternions, therefore, the two formulas give different results, the first being the usual form for matrix multiplication, but the second formula enables us to represent octonions by using 2×2 matrices over the quaternions. An octonion can then be written as the 2×2 matrix with the special form,

$$o = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} \tag{3}$$

where the A^* and B^* are quaternion conjugates of the corresponding A and B quaternions. The octonions have 8 degrees of freedom, and are thus a subset of that more general number represented by the 2×2 matrices of quaternion entries. These special hexadecanions, which include the multiplication formula (2), we shall call "*quatro-quaternions.*"

$$\text{R}(A, B) = A \cdot B$$
$$\text{L}(A, B) = B \cdot A$$

Since quaternion products non-commute, this means that we have to discriminate between RIGHT and LEFT actions when considering products. Such consideration is not necessary for reals and complex numbers, but for quaternions the issue becomes important. The concepts of right and left are *relative* to each other. But, if, for a *right-action* product, we write, $\text{R}(A, B) = A \cdot B$, then there is always a related product, $\text{L}(A, B) = B \cdot A$, which is *left-action.* Our two matrix product formulas are then distinguished by the way in which they combine these *right* and *left* actions.

$$\begin{pmatrix} \text{R+R} & \text{R+R} \\ \text{R+R} & \text{R+R} \end{pmatrix} \qquad \begin{pmatrix} \text{R+L} & \text{L+R} \\ \text{L+R} & \text{R+L} \end{pmatrix}$$

In the standard matrix product all the matrix entries are, R+R, that is, they are formed by summing right action products of quaternions. While, in the new special matrix product, there is a balance, R+L, between right and left actions in the sum. This latter combination alternates between R+L and L+R on moving to the next column or row.

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When the matrix entries have products that associate and commute, both these matrix products, $\cdot$ and $\times$, also associate, and indeed produce the same results. But, when the matrix entries have products that non-commute, these *matrix product operators* become distinct; the $\cdot$ remains "*associative*," but the $\times$ is now "*non-associative*."

Here the *commutativity* of the underlying number system determines the *associativity* of the higher order number system erected over those underlying numbers. This concept of the " *non-associative product* " can be extended to matrices of all orders, by replacing the usual right-action summations, R+R+R+... , with appropriate balanced alternating right and left corresponding sums, like R+L+R+... . But, for the moment, our interest is in 2×2 matrices. Even though the second type of matrix product is only non-associative when the underlying is non-commutative, we shall still generally refer to this type of product as the *non-associative product*, to distinguish the *form* from the standard product. A better name might be the " *interleaved product*, " since the right and left actions are really being interleaved, or interweaved, in the summation. Other names might be *twisted product*, *string product*, *chain product*, *braided product*, and so on, given that the alternating form of the right and left actions weave in and out like the twisted action found in the braiding of a string, rope, or chain, since the form of the summation goes like $A \cdot B + B \cdot A + A \cdot B + B \cdot A + ...$ etc. The name we establish for this type of product should be unique from other names already being used for different mathematical concepts, so we provide these various suggestions. However, we shall often find the name " *twisted product* " very convenient, as we shall see elsewhere in this paper. Our general matrix algebra then, now has two types of multiplications, one associative and one non-associative. When we restrict the product to `type one`—the standard multiplication—we have $M(2, \mathbb{H})$, which we also write, $M_{[\cdot]}(2, \mathbb{H})$, the standard algebra of 2×2 matrices over Hamilton's quaternions. When we restrict the product to `type two`—the new multiplication—we have "*restricted quatro-quaternions*," which we symbolize, $M_{[\times]}(2, \mathbb{H})$, and which is also a 2×2 matrix algebra over Hamilton's quaternions, except with the new special product definition replacing the standard matrix product. Octonions are then a natural subalgebra of these restricted quatro-quaternions, $\mathbb{O} \subset M_{[\times]}(2, \mathbb{H})$, and also, $\mathbb{H} \subset M_{[\times]}(2, \mathbb{C})$, and $\mathbb{C} \subset M_{[\times]}(2, \mathbb{R})$, but there being no significant difference between the choice of product, $\cdot$ or $\times$, in these latter two cases, we can just continue to write, $\mathbb{H} \subset M(2, \mathbb{C})$, and $\mathbb{C} \subset M(2, \mathbb{R})$, as usual. Without these restrictions, we have an entirely new type of matrix algebra with three essential operators $\{+, \cdot, \times\}$, one addition and two multiplications, which we call "*quatro-quaternions*," symbolized by $\mathbb{QQ} \equiv M_{[\cdot,\times]}(2, \mathbb{H})$.

Defining such an *interleaved product* is simpler for 2×2 matrices than $n \times n$. However, even with just 2×2 matrices to deal with, once we recognise that combinations of right and left actions empower us to extend the matrix algebra to get greater flexibility in algebraic representations, we find we still have very many alternative ways to do this. Since there are 4 matrix entries, each formed from the sum of 2 quaternion pair products, like R+R, or R+L, and there are thus 8 terms within a matrix product that can take the R or L form for the action, we have $2^8 = 256$ ways to define a matrix product. So, technically, we could construct 256 different matrix algebras, each based on one of these particular choices for product. The standard matrix product uses up one of these possibilities, where all the actions are R. But, that still leaves 255 alternative forms for us to choose from in defining our alternate product. The extension we chose is inspired by the Cayley-Dickson construction. So, it is probably appropriate to review this construction here. There are two general ways to define the Cayley-Dickson construction, but they both lead to the same essential results.

CAYLEY-DICKSON (I) : $(A, B)^* \equiv (A^*, -B)$

$$(A, B)(C, D) = (AC - DB^*, A^*D + CB) \tag{4}$$

$$o = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} \qquad\qquad o^* = \begin{pmatrix} A^* & B^* \\ -B & A \end{pmatrix} \tag{5}$$

CAYLEY-DICKSON (II) : $(A, B)^* \equiv (A^*, -B)$

$$(A, B)(C, D) = (AC - D^*B, DA + BC^*) \tag{6}$$

$$o = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix} \qquad\qquad o^* = \begin{pmatrix} A^* & -B \\ B^* & A \end{pmatrix} \tag{7}$$

They derive from setting, $(A, B) = A + iB$, with an i prefix, and, $(A, B) = A + Bi$, with an i postfix, respectively. The higher dimensional algebra can also be represented by 2×2 matrices over the next lower dimensional algebra, using a pair of elements from that lower algebra as shown in (5) and (7). When A, B are reals, o is a complex number. When A, B are complex numbers, o is a quaternion. And when A, B are quaternions, o is an octonion, provided we use the non-standard matrix product defined in (2). Apart from this last case, it doesn't matter which matrix product we use, (1) or (2), both give us the same way to represent complex numbers or quaternions by 2×2 matrices. Our new *interleaved product* works with either form of the Cayley-Dickson construction. Note, however, that the 2×2 matrix definitions of the o's are transposed when changing construction from (4) to (6). It is instructive to follow the inductive steps that lead to these two forms for the Cayley-Dickson construction.

Using $(A, B) = A + iB$,

$$
\begin{aligned}
(A,B)(C,D) &= (A+iB)(C+iD) && (8)\\
&= AC + iBiD + AiD + iBC && (9)\\
&= AC + i^2B^*D + iA^*D + iBC && (10)\\
&= (AC - B^*D) + i(A^*D + BC) && (11)\\
&= (AC - B^*D, A^*D + BC) && (12)\\
&\approx (\,\mathrm{R} - \mathrm{R}\,,\ \mathrm{R} + \mathrm{R}\,) && (13)\\
&\texttt{...\ twisted\ into\ ...}\\
&= (AC - DB^*, A^*D + CB) && (14)\\
&\approx (\,\mathrm{R} - \mathrm{L}\,,\ \mathrm{R} + \mathrm{L}\,) && (15)
\end{aligned}
$$

Using $(A, B) = A + Bi$,

$$
\begin{aligned}
(A,B)(C,D) &= (A+Bi)(C+Di) && (16)\\
&= AC + BiDi + ADi + BiC && (17)\\
&= AC + BD^*i^2 + ADi + BC^*i && (18)\\
&= (AC - BD^*) + (AD + BC^*)i && (19)\\
&= (AC - BD^*, AD + BC^*) && (20)\\
&\approx (\,\mathrm{R} - \mathrm{R}\,,\ \mathrm{R} + \mathrm{R}\,) && (21)\\
&\texttt{...\ twisted\ into\ ...}\\
&= (AC - D^*B, DA + BC^*) && (22)\\
&\approx (\,\mathrm{R} - \mathrm{L}\,,\ \mathrm{L} + \mathrm{R}\,) && (23)
\end{aligned}
$$

We arrive at the idea of the Cayley-Dickson construction by observing how the introduction of a new imaginary element, i, into an already existing hypercomplex number, induces the doubling of the dimension. Then following Hamilton's method of the algebra of couples, we represent this double number by an ordered pair, (A, B), which allows us to emphasize this doubling without making explicit reference to a new imaginary element, like, i, indeed without making any reference to imaginary quantities at all, since all numbers are ultimately built from pairs of reals.

Starting with real numbers, we double them up by introducing our first imaginary, $(A, B) = A + iB$. Multiplication is then written, $(A, B)(C, D) = (A+iB)(C+iD) = AC+iBiD+AiD+iBC$. Our imaginary element, i, commutes and associates with the reals under multiplication, and, $i^2 = -1$, by definition, so we get $(AC - BD, AD + BC)$, and the conjugate of our complex number is $(A, B)^* = (A, -B)$. When we double these complex numbers, we have one type of modification to make to this product formula. Our new imaginary, i, is orthogonal to the existing imaginary element that defines the complex number, and anticommutes with the old imaginary element, by definition, and for every complex number, A, the new i conjugates the parameter when we commute the variables, i.e. $Ai = iA^*$, $iA = A^*i$. So, we have to introduce a modification to account for this conjugation, and the steps above show how we obtain two forms (12) or (20) depending on whether we use the form $A + iB$ or $A + Bi$ to derive our ordered couple formula. The conjugate of the new double number now becomes, $(A, B)^* = (A^*, -B)$. The introduction of the conjugation * in appropriate places, is the only modification we need to make, and since real numbers are self-conjugate, the new formulas are valid for constructing complex numbers from reals, as well as constructing quaternions from complex numbers. But now, when we come to double the quaternions to obtain our octonions, there's one new modification we have to make to obtain the correct formulas. Since quaternions don't commute under multiplication, we have to discriminate between RIGHT and LEFT action products, $\mathrm{R}(A, B) = AB$ and $\mathrm{L}(A, B) = BA$. The order of the factors is now an important consideration, and they happen not to be in proper order here, but a simple *twisting* does the trick.

Why do we need to twist?

(P1): $N(a) = |A|^2 + |B|^2 = |a|^2 \geq 0.$ $\qquad a = (A, B)$
(P2): $N(a) = a \cdot a^* = a^* \cdot a.$ $\qquad b = (C, D)$
(P3): $N(a)N(b) = N(ab).$

We'd like our algebra to have a "norm", $N(\cdot)$, with as many of the usual properties exhibited by the previous algebras as possible. *First*, we'd like there to exist a formula, so that for each number, a, in our algebra, $N(a)$ produces a positive real value for non-zero a numbers, vanishing only when the number a itself vanishes, which we can therefore always write, $N(a) = |a|^2$. Then, we'd like this norm formula to be constructed using all the component parts of the number a, so that it can be indeed constructed to vanish only when a itself vanishes. We'd like this formula to be the "sum of squares" of the component parts of the number a, so that, if, $a = (A, B)$, we may write, $N(a) = |A|^2 + |B|^2$. And, given that these parts, A and B, are themselves from such normed algebras of lower dimension, the norm can be ultimately expanded into the "sum of n squares" of the real components in the n-tuple, $a = (a_0, a_1, \ldots, a_{n-1})$, $N(a) = a_0^2 + a_1^2 + \ldots + a_{n-1}^2$, thus covering all the degrees of freedom in the n-dimensional a-number. With this alone established, we'll refer to the construction as a property (P1) norm. We might be interested in what transformations on our a-numbers keep this norm invariant. In which case, it's simply a "metric" that describes some geometric feature of the algebra. *Second,* we'd like to be able to construct multiplicative inverses for the a numbers using the norm—after all, it would be nice to actually be able to do something algebraically with our norm, instead of just using it to paint pretty geometric pictures. The usual way to exploit the norm is in its natural link to the concept of the conjugate exhibited in the lower dimensional algebras. So, first we must require our algebra have such an operation as conjugation, and that conjugation must have certain nice properties. One convention is to define an algebra, $\mathcal{A}$, to be a `*-algebra`, if there's a real-linear map, $* : \mathcal{A} \mapsto \mathcal{A}$, i.e. $(\lambda a + \lambda b)^* = \lambda a^* + \lambda b^*$, $\lambda \in \mathbb{R}$, which obeys the two rules, $(a^*)^* = a$, $(ab)^* = b^* a^*$, $\forall\, a, b, \in \mathcal{A}$. If, additionally, $\forall\, a \in \mathcal{A} : a + a^*, a \cdot a^*, a^* \cdot a \in \mathbb{R}$, and also, $\forall\, a \neq 0 :\ a \cdot a^* = a^* \cdot a > 0$, then we can define a norm, $|a|^2 = a \cdot a^* = a^* \cdot a$, and now use it to construct the multiplicative inverse, $a^{-1} = a^*/|a|^2$, so the standard practice is to then call this `*-algebra` *nicely-normed* and be content we got this far. With this established, therefore, we'll refer to the construction as a property (P2) norm. Now we can use our norm formula to construct inverses. *Third*, we'd like the product of norms from two different numbers to equal the norm of their product. This is usually accomplished through the "law of the squares" which tells us that the product of two sums of n squares is a sum of n squares, and is a particularly useful feature, e.g. combined with (P2) we could write, $(ab)^{-1} = b^{-1}a^{-1}$. With this established, we'll refer to the construction as a $(P3)$ norm.

Now, it's a simple matter to construct a (P1) norm. We just declare that, for any number, $a = (A, B)$, we define, $N(a) = |A|^2 + |B|^2$, using the properties of the previous norms to establish our norm. But, now we'd like our norm to satisfy the (P2) property. We know that the formula for the conjugate is, $a^* = (A^*, -B)$, from the previous quaternion construction, and so we start with this definition also for our octonion conjugate, not knowing yet what, if anything, needs modification. Then, using the expression (12) for the product, we test the (P2) property,

$$N(a) = (A, B) \cdot (A, B)^* = (A, B) \cdot (A^*, -B) = (AA^* - B^*(-B), A^*(-B) + BA^*) \tag{24}$$

$$= (\ |A|^2 + |B|^2\ ,\ -A^*B + BA^*\ ) \tag{25}$$

`so we try...`

$$(A, B) \cdot (C, D) = (AC - B^*D, A^*D + CB) \tag{26}$$

and we notice that we can obtain the norm property, by keeping the previous definition of the conjugate, if we get the terms on the right side of the resultant ordered couple to cancel out. This can be accomplished by reversing factors in either pair product, A^*D or BC, in the right half of the couple in definition (12). We must *twist one* of these terms, but not both simultaneously. The relative order is what matters. Either one we pick will give us the results we seek. Say then, we replace BC with CB. Then, our product definition (12) is replaced by (26), and result (25) becomes, $N(a) = (|A|^2 + |B|^2) \cdot (1, 0) = |A|^2 + |B|^2$, where we identify the ordered couple, $(1, 0)$, with the real scalar unit 1.

Since we'd like our octonions to satisfy (P3) also, we start with this new twisted product (26) and test the property,

$$N(ab) = (AC - B^*D, A^*D + CB) \cdot (AC - B^*D, A^*D + CB)^*$$

$$= (\ (|A|^2 + |B|^2)(|C|^2 + |D|^2) - 2.S(ACD^*B) + 2.S(D^*ACB)\ ,\ 0\ ) \tag{27}$$

Unfortunately, although we find the right result, $N(a)N(b)$, in part of this expression, we also have another part that messes things up. The notation, $S(\cdot)$, refers to the scalar part of the quaternion. We may cyclically permute the quaternion factors within when taking the scalar, so, $S(D^*ACB) = S(ACBD^*)$, for example, but this is the closest we can get to the other term, $S(ACD^*B)$. We still differ by, BD^* verses D^*B. The solution is to effect a another twist.

In order to get the most useful normed algebra, we have to alter the order of the factors in two of the terms. This means that some *right action* products are replaced by their corresponding *left action* products. All products are not changed, just one term in the sum on each side of the ordered couple. This thus results in a *twisted product* formula for the Cayley-Dickson construction, where sums of right action products, like R-R and R+R, are replaced by sums that balance right and left actions, like R-L and R+L or L+R. This is the very *twisting action* required in the modification of the matrix product, in order to enable the matrix algebra to also represent octonions in terms of 2×2 matrices over the quaternions. Note that this twisting modification has no effect on the previous hypercomplex numbers, since reals and complex numbers commute. The issue only becomes important when doubling the quaternions to get octonions. So, the new twisted product formulas work just as well for the lower dimensional numbers where right and left actions are indistinguishable from one another. The Cayley-Dickson construction is typically fixed at this final conjugated and twisted formula, i.e. (14) or (22), since no obvious advantage is obtained by further modifications[1], there being only four normed division algebras. Beyond the octonions, one can still continue to double the numbers using this final formula. The Cayley-Dickson algebras that result are all nicely normed and power associative, but are not division algebras, not associative, nor alternative, and, of course, they don't commute. One can, however, continue to use the new matrix product to represent these Cayley-Dickson algebras by a matrix algebra.

II. QUATRO-QUATERNIONS.

While any Cayley-Dickson algebra can be represented by 2×2 matrices over the previous algebra, with this new non-associative matrix product, actual calculation with the new 2×2 matrix representation is non-trivial. Since we've changed the very definition of the product, we need to re-construct the formula for the multiplicative inverse of a matrix implied by this new *twisted product.* In the case of quatro-quaternions, at least, we can do this easily, using our knowledge of two-hand quaternions. Consider the following matrix equation,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \times \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{28}$$

If the As and Bs are right-hand quaternions, then our special *twisted product* formula (2) gives us two sets of systems of linear quaternion equations to solve.

$$\begin{aligned} A_{00}B_{00} + B_{10}A_{01} &= 1 & \qquad B_{01}A_{00} + A_{01}B_{11} &= 0 \\ B_{00}A_{10} + A_{11}B_{10} &= 0 & \qquad A_{10}B_{01} + B_{11}A_{11} &= 1 \end{aligned} \tag{29}$$

A special method to solve these quaternion systems is described in detail in a previous paper [PJ2] [2]. Essentially, we can move the known factors over to the other side of the unknowns, by converting right-handed quaternion factors into their left-handed quaternion forms. If, therefore, we take the matrix with the As to be the known parameters, and the matrix with the Bs to be the unknowns, we can move those As that stand on the R-H-S of the Bs, over to the L-H-S, by changing the moving As into left-hand quaternions, which we indicate here with the $'$ mark, e.g. $A_{01} \mapsto A'_{01}$, and simultaneously, we mark the unknowns with a caret $\hat{}$ to indicate which parameters are being used as pivots, e.g. $B_{10} \mapsto \hat{B}_{10}$. This allows us to arrange all the knowns on one side of the unknowns.

$$\begin{aligned} A_{00}\hat{B}_{00} + A'_{01}\hat{B}_{10} &= \hat{1} & \qquad A'_{00}\hat{B}_{01} + A_{01}\hat{B}_{11} &= \hat{0} \\ A'_{10}\hat{B}_{00} + A_{11}\hat{B}_{10} &= \hat{0} & \qquad A_{10}\hat{B}_{01} + A'_{11}\hat{B}_{11} &= \hat{1} \end{aligned} \tag{30}$$

In effect, we have " `un-twisted` " the product formulas, by making use of the two-hand quaternion algebra techniques. Non-abelian algebra presents us with many such twisted product forms that make manipulation difficult. But, fortunately, when dealing with quaternions, we have these special methods to un-twist the expressions, which then allow us to work with the forms as if we're dealing with a familiar abelian algebra. Then, using the fact that left-handed quaternions commute with right-handed quaternions, the usual methods of re-arrangement allow these equations to be re-written as follows, each in terms of one unknown variable.

$$\begin{aligned} A_{11}A_{00}\hat{B}_{00} - A'_{01}A'_{10}\hat{B}_{00} &= A_{11}\hat{1} & \qquad A_{01}A_{10}\hat{B}_{01} - A'_{11}A'_{00}\hat{B}_{01} &= A_{01}\hat{1} \\ A'_{10}A'_{01}\hat{B}_{10} - A_{00}A_{11}\hat{B}_{10} &= A'_{10}\hat{1} & \qquad A'_{00}A'_{11}\hat{B}_{11} - A_{10}A_{01}\hat{B}_{11} &= A'_{00}\hat{1} \end{aligned} \tag{31}$$

The solution to the matrix with the Bs is then given by the four parameter results[2];

$$\begin{aligned} \hat{B}_{00} &= \frac{A_{11}\hat{1}}{\vdash (A_{11}A_{00} - A'_{01}A'_{10})} & \qquad \hat{B}_{01} &= \frac{A_{01}\hat{1}}{\vdash (A_{01}A_{10} - A'_{11}A'_{00})} \\ \hat{B}_{10} &= \frac{A'_{10}\hat{1}}{\vdash (A'_{10}A'_{01} - A_{00}A_{11})} & \qquad \hat{B}_{11} &= \frac{A'_{00}\hat{1}}{\vdash (A'_{00}A'_{11} - A_{10}A_{01})} \end{aligned} \tag{32}$$

The formulas in (32) make use of the hand changing operator $'$, which has some similarities to the more familiar conjugation operator *. For example, like $(AB)^* = B^*A^*$, the hand change obeys a similar rule, $(AB)' = B'A'$, in that the factors are reversed in the product when removing the parenthesis; and like, $(A^*)^* = A$, we have, $(A')' = A$. We could use this, for example, to write $(A'_{00}A'_{11} - A_{10}A_{01}) = (A_{11}A_{00} - A'_{01}A'_{10})'$, etc., and thus show that the four dividing factors in the expressions for the Bs are really just two different factors and their hand-changed versions, i.e.,

$$\begin{pmatrix} \hat{B}_{00} & \hat{B}_{01} \\ \hat{B}_{10} & \hat{B}_{11} \end{pmatrix} = \begin{pmatrix} \frac{A_{11}\hat{1}}{\vdash D_1} & -\frac{A_{01}\hat{1}}{\vdash D_3} \\ -\frac{A'_{10}\hat{1}}{\vdash D_2} & \frac{A'_{00}\hat{1}}{\vdash D_4} \end{pmatrix} = \begin{pmatrix} \frac{A_{11}\hat{1}}{\vdash D_1} & -\frac{A_{01}\hat{1}}{\vdash D'_2} \\ -\frac{A'_{10}\hat{1}}{\vdash D_2} & \frac{A'_{00}\hat{1}}{\vdash D'_1} \end{pmatrix} \tag{33}$$

`where,`

$$D_1 = A_{11}A_{00} - A'_{01}A'_{10}\ , \quad D_2 = A_{00}A_{11} - A'_{10}A'_{01}\ , \quad D_3 = D'_2\ , \quad D_4 = D'_1\ .$$

If the As and Bs were abelian factors, and we were thus dealing with the ordinary matrix algebra over these abelian parameters, we'd have a much simpler expression for the matrix inverse. The B-matrix would then take the form,

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} \frac{A_{11}}{D} & -\frac{A_{01}}{D} \\ -\frac{A_{10}}{D} & \frac{A_{00}}{D} \end{pmatrix} = \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{D} \end{pmatrix} \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} = \frac{1}{D} \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} \tag{34}$$

`where,`

$$\begin{array}{llll} & D = \begin{vmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{vmatrix} & = A_{00}A_{11} - A_{01}A_{10} & \approx \quad \mathrm{R-R} \\ \texttt{cf.} & D_1 & = A_{11}A_{00} - A'_{01}A'_{10} & \approx \quad \mathrm{L-R} \\ & D_2 & = A_{00}A_{11} - A'_{10}A'_{01} & \approx \quad \mathrm{R-L} \\ & D_3 & = A'_{11}A'_{00} - A_{01}A_{10} & \approx \quad \mathrm{L-R} \\ & D_4 & = A'_{00}A'_{11} - A_{10}A_{01} & \approx \quad \mathrm{R-L} \end{array} \tag{35}$$

Like the Cayley-Dickson construction, which adds *twisting and conjugation* to complete the product formulas, here our new matrix inverse construction adds *twisting and hand transformation* to arrive at the required corresponding modified forms. Our single determinant factor, in the ordinary matrix algebra, is now split into four different forms. Comparing the single determinant to the four new divisors, we see that characteristic twisting profile alteration balancing the right and left actions once again, and then, of course, we have hand changes in addition to this twisting.

THE STANDARD $\cdot$ PRODUCT:

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \cdot \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{36}$$

Now, when we're dealing with the standard matrix product (36), and the As and Bs are our non-abelian quaternions, we start with systems of linear quaternion equations with product expressions already un-twisted. In fact, by simply removing the primes $'$ and hats $\hat{}$ from the equations (30) we'd have our starting point,

$$\begin{array}{ll} A_{00}B_{00} + A_{01}B_{10} = 1 & A_{00}B_{01} + A_{01}B_{11} = 0 \\ A_{10}B_{00} + A_{11}B_{10} = 0 & A_{10}B_{01} + A_{11}B_{11} = 1 \end{array} \tag{37}$$

The procedure for re-arranging these equations, however, is a little different. Right handed quaternions don't commute with each other, so we can't simplify the expressions by multiplying by these same A-factors. Instead, we must multiply by the inverses of these known parameters. Then we can re-write these equations, each in terms of one unknown,

$$\begin{array}{ll} A_{01}^{-1}A_{00}B_{00} - A_{11}^{-1}A_{10}B_{00} = A_{01}^{-1} & A_{11}^{-1}A_{10}B_{01} - A_{01}^{-1}A_{00}B_{01} = A_{11}^{-1} \\ A_{00}^{-1}A_{01}B_{10} - A_{10}^{-1}A_{11}B_{10} = A_{00}^{-1} & A_{10}^{-1}A_{11}B_{11} - A_{00}^{-1}A_{01}B_{11} = A_{10}^{-1} \end{array} \tag{38}$$

which then leads to the solution,

$$\begin{array}{ll} B_{00} = \dfrac{A_{01}^{-1}}{\vdash (A_{01}^{-1}A_{00} - A_{11}^{-1}A_{10})} & B_{01} = \dfrac{A_{11}^{-1}}{\vdash (A_{11}^{-1}A_{10} - A_{01}^{-1}A_{00})} \\ B_{10} = \dfrac{A_{00}^{-1}}{\vdash (A_{00}^{-1}A_{01} - A_{10}^{-1}A_{11})} & B_{11} = \dfrac{A_{10}^{-1}}{\vdash (A_{10}^{-1}A_{11} - A_{00}^{-1}A_{01})} \end{array} \tag{39}$$

we can write this in terms of two dividing factors, D_1 and D_2,

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} -\frac{A_{01}^{-1}}{\vdash D_1} & \frac{A_{11}^{-1}}{\vdash D_1} \\ \frac{A_{00}^{-1}}{\vdash D_2} & -\frac{A_{10}^{-1}}{\vdash D_2} \end{pmatrix} = \begin{pmatrix} \frac{1}{D_1} & 0 \\ 0 & \frac{1}{D_2} \end{pmatrix} \cdot \begin{pmatrix} -A_{01}^{-1} & A_{11}^{-1} \\ A_{00}^{-1} & -A_{10}^{-1} \end{pmatrix} \tag{40}$$

where,

$$D_1 = A_{11}^{-1} A_{10} - A_{01}^{-1} A_{00} \qquad D_2 = A_{00}^{-1} A_{01} - A_{10}^{-1} A_{11}$$

but, with all the inverse parameters in this expression, it's hard to compare to our previous formulas for the ordinary matrix inverse (34), or the twisted product inverse (33). Let us then re-express the equations (38). We can also write,

$$\begin{aligned} A_{11}A_{00}B_{00} - A_{11}A_{01}A_{11}^{-1}A_{10}B_{00} &= A_{11} \qquad & A_{01}A_{10}B_{01} - A_{01}A_{11}A_{01}^{-1}A_{00}B_{01} &= A_{01} \\ A_{10}A_{01}B_{10} - A_{10}A_{00}A_{10}^{-1}A_{11}B_{10} &= A_{10} \qquad & A_{00}A_{11}B_{11} - A_{00}A_{10}A_{00}^{-1}A_{01}B_{11} &= A_{00} \end{aligned} \tag{41}$$

which now gives us a more familiar form,

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} \frac{A_{11}}{\vdash D_1} & -\frac{A_{01}}{\vdash D_3} \\ -\frac{A_{10}}{\vdash D_2} & \frac{A_{00}}{\vdash D_4} \end{pmatrix} \tag{42}$$

where,

$$\begin{aligned} D_1 &= (A_{11}A_{00} - A_{11}A_{01}A_{11}^{-1}A_{10}), \qquad & D_2 &= (A_{10}A_{00}A_{10}^{-1}A_{11} - A_{10}A_{01}), \\ D_4 &= (A_{00}A_{11} - A_{00}A_{10}A_{00}^{-1}A_{01}), \qquad & D_3 &= (A_{01}A_{11}A_{01}^{-1}A_{00} - A_{01}A_{10}), \end{aligned}$$

The inverse solutions (40) and (42) are equivalent. But, in the latter we have four divisors, D_μ, $\mu = 1, 2, 3, 4$, and these are not easily re-expressed in terms of just 2, the way in which they are found in the first expression. However, the numerators and signs are the familiar ones from our other inverse matrix formulas, and this allows us to see just what comparative modifications are required in the standard product case in contrast to the twisted product.

These results, (40) and (42), are expressions for the right side inverse for the associative $\cdot$ product, while (33) gives the corresponding right side inverse for the non-associative $\times$ product. To get the left-inverses, we now take the As to be our unknowns, and let the Bs be our knowns, and re-solve these equations in a similar manner.

For the non-associative $\times$ product we obtain the following left inverse;

$$\begin{aligned} \hat{A}_{00} &= \frac{B'_{11}\hat{1}}{\vdash (B'_{11}B'_{00} - B_{10}B_{01})} \qquad & \hat{A}_{01} &= \frac{B_{01}\hat{1}}{\vdash (B_{01}B_{10} - B'_{00}B'_{11})} \\ \hat{A}_{10} &= \frac{B'_{10}\hat{1}}{\vdash (B'_{10}B'_{01} - B_{11}B_{00})} \qquad & \hat{A}_{11} &= \frac{B_{00}\hat{1}}{\vdash (B_{00}B_{11} - B'_{01}B'_{10})} \end{aligned} \tag{43}$$

$$\begin{pmatrix} \hat{A}_{00} & \hat{A}_{01} \\ \hat{A}_{10} & \hat{A}_{11} \end{pmatrix} = \begin{pmatrix} \frac{B'_{11}\hat{1}}{\vdash D_1} & -\frac{B_{01}\hat{1}}{\vdash D_3} \\ -\frac{B'_{10}\hat{1}}{\vdash D_2} & \frac{B_{00}\hat{1}}{\vdash D_4} \end{pmatrix} = \begin{pmatrix} \frac{B'_{11}\hat{1}}{\vdash D_1} & -\frac{B_{01}\hat{1}}{\vdash D'_2} \\ -\frac{B'_{10}\hat{1}}{\vdash D_2} & \frac{B_{00}\hat{1}}{\vdash D'_1} \end{pmatrix} \tag{44}$$

where,

$$D_1 = B'_{11}B'_{00} - B_{10}B_{01}\ , \quad D_2 = B_{11}B_{00} - B'_{10}B'_{01}\ , \quad D_3 = D'_2\ , \quad D_4 = D'_1\ .$$

where we have solved for these expressions working from the same side of the equations as we did before to obtain the corresponding (33) right-side inverse results, that is, we again move all the unknowns over to the L.H.S of the knowns. This allows us to write our formulas with the divisors appearing on the same side as before, so that we can compare the expressions more easily. A consequence of this approach, is that when comparing our divisors to the previous results in (35) we find they now have the unbalanced twisted product forms, $D_1 \approx$ L$-$L, $D_3 \approx$ R$-$R.

However, looking back at our original comparisons in (35), we notice that the determinant in ordinary matrix algebra really doesn't have a definitive twisted product form. In fact, it relies on the permutation symmetry in the multiplication of factors to manifest as just one divisor, D, instead of four.

The twisted profiles, R - R , R - L , L - R , L - L , are all equivalent in ordinary matrix algebra, and any one of these could be picked to set our reference point for D. We set the reference to R - R to illustrate how the comparative products for, D_μ, $\mu = 1, 2, 3, 4$, could be considered twisted modifications of the original state. But, if we either started with a different reference, or solved for these formulas from the other side of the equations, our divisors would reflect a different twisting relative to our reference. Whatever the case, however, we always get either the balanced pair R - L and L - R , or the unblanced pair, R - R and L - L , in our four divisors.

For the left inverse of the standard $\cdot$ product, we can't continue to solve from the same side of the equations, as we have done repeatedly above (unless we want to use the two-hand quaternion method for this inverse, which doesn't really require it), so we solve (37) from the other side of the equations this time. Our divisors are now on the right, symbolized by $\dashv$ instead of the previous $\vdash$ in our denominators. We obtain the following,

$$A_{00} = \frac{B_{10}^{-1}}{(B_{00}B_{10}^{-1} - B_{01}B_{11}^{-1}) \dashv} \qquad A_{01} = \frac{B_{00}^{-1}}{(B_{10}B_{00}^{-1} - B_{11}B_{01}^{-1}) \dashv}$$
$$A_{10} = \frac{B_{11}^{-1}}{(B_{01}B_{11}^{-1} - B_{00}B_{10}^{-1}) \dashv} \qquad A_{11} = \frac{B_{01}^{-1}}{(B_{11}B_{01}^{-1} - B_{10}B_{00}^{-1}) \dashv} \tag{45}$$

we can write this in terms of two dividing factors, D_1 and D_2,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} -\dfrac{B_{10}^{-1}}{D_1 \dashv} & \dfrac{B_{00}^{-1}}{D_2 \dashv} \\ \\ \dfrac{B_{11}^{-1}}{D_1 \dashv} & -\dfrac{B_{01}^{-1}}{D_2 \dashv} \end{pmatrix} = \begin{pmatrix} -B_{10}^{-1} & B_{00}^{-1} \\ \\ B_{11}^{-1} & -B_{01}^{-1} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{D_1} & 0 \\ 0 & \frac{1}{D_2} \end{pmatrix} \tag{46}$$

where,

$$D_1 = B_{01}B_{11}^{-1} - B_{00}B_{10}^{-1} \qquad D_2 = B_{10}B_{00}^{-1} - B_{11}B_{01}^{-1}$$

Notice that the diagonal matrix of divisors has moved to the right. This is the left-inverse corresponding to the right-inverse (40). But again, with all the inverse parameters in this expression, it's hard to compare to our previous formulas for the ordinary matrix inverse (34), or the twisted product inverse (33). So, we once again re-express this result in a more familiar form,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{B_{11}}{D_1 \dashv} & -\dfrac{B_{01}}{D_3 \dashv} \\ \\ -\dfrac{B_{10}}{D_2 \dashv} & \dfrac{B_{00}}{D_4 \dashv} \end{pmatrix} \tag{47}$$

where,

$$D_1 = (B_{00}B_{11} - B_{01}B_{11}^{-1}B_{10}B_{11}), \qquad D_2 = (B_{00}B_{10}^{-1}B_{11}B_{10} - B_{01}B_{10}),$$
$$D_4 = (B_{11}B_{00} - B_{10}B_{00}^{-1}B_{01}B_{00}), \qquad D_3 = (B_{11}B_{01}^{-1}B_{00}B_{01} - B_{10}B_{01}),$$

and can compare this left-inverse (47) to the corresponding right-inverse (42). When seeking to inspect the differences among the various inverses, these forms help to reveal what's changed. However, in the case of the $\cdot$ product, we have yet another way to write the inverse formulas. This time all the numerators are 1, so left and right divisions are the same, we no longer need the $\vdash$ and $\dashv$ symbols. This allows us to see that once a matrix has a left inverse it has a right inverse, and visa versa, and the two are always the same; a fact hidden in the complexity of the other formats.

LEFT $\cdot$ INVERSE $A[..] =$ | RIGHT $\cdot$ INVERSE $B[..] =$

$$\begin{pmatrix} \dfrac{1}{(B_{00} - B_{01}B_{11}^{-1}B_{10})} & \dfrac{1}{(B_{10} - B_{11}B_{01}^{-1}B_{00})} \\ \\ \dfrac{1}{(B_{01} - B_{00}B_{10}^{-1}B_{11})} & \dfrac{1}{(B_{11} - B_{10}B_{00}^{-1}B_{01})} \end{pmatrix} \quad \Big| \quad \begin{pmatrix} \dfrac{1}{(A_{00} - A_{01}A_{11}^{-1}A_{10})} & \dfrac{1}{(A_{10} - A_{11}A_{01}^{-1}A_{00})} \\ \\ \dfrac{1}{(A_{01} - A_{00}A_{10}^{-1}A_{11})} & \dfrac{1}{(A_{11} - A_{10}A_{00}^{-1}A_{01})} \end{pmatrix} \tag{48}$$

A similar comparison for the left and right $\times$ product inverses reveals that they are generally different;

$$\text{LEFT} \times \text{INVERSE } \hat{A}[..] = \qquad \qquad \text{RIGHT} \times \text{INVERSE } \hat{B}[..] =$$

$$\begin{pmatrix} \frac{\hat{1}}{\vdash (B'_{00} - B'^{-1}_{11} B_{10} B_{01})} & \frac{\hat{1}}{\vdash (B_{10} - B^{-1}_{01} B'_{00} B'_{11})} \\ \frac{\hat{1}}{\vdash (B'_{01} - B'^{-1}_{10} B_{11} B_{00})} & \frac{\hat{1}}{\vdash (B_{11} - B^{-1}_{00} B'_{01} B'_{10})} \end{pmatrix} \quad | \quad \begin{pmatrix} \frac{\hat{1}}{\vdash (A_{00} - A^{-1}_{11} A'_{01} A'_{10})} & \frac{\hat{1}}{\vdash (A_{10} - A^{-1}_{01} A'_{11} A'_{00})} \\ \frac{\hat{1}}{\vdash (A'_{01} - A'^{-1}_{10} A_{00} A_{11})} & \frac{\hat{1}}{\vdash (A'_{11} - A'^{-1}_{00} A_{10} A_{01})} \end{pmatrix} \tag{49}$$

But, by inspection we can see that the corresponding divisors on the main diagonal are just hand transforms of each other, e.g. $(B'_{00} - B'^{-1}_{11} B_{10} B_{01})' = (B'_{00})' - (B'^{-1}_{11} B_{10} B_{01})' = B_{00} - B'_{01} B'_{10} B^{-1}_{11}$, and since the right hand quaternion B^{-1}_{11} commutes with the left hand quaternions, B'_{10} and B'_{01}, we can permute the factors to obtain, $B_{00} - B^{-1}_{11} B'_{01} B'_{10}$, which is then identical in form to the corresponding divisor, $A_{00} - A^{-1}_{11} A'_{01} A'_{10}$, on the other side. Now, if $X' = Y$, and $Y = 0$, then $X = 0$, also; i.e. the hand transform of 0 is always 0. So, whenever one of the divisors on the main diagonal vanishes, neither left nor right inverses exist. Now, if we examine when the top element on the cross diagonal vanishes, i.e. $B_{10} - B^{-1}_{01} B'_{00} B'_{11} = 0$, we see the condition is, $B_{01} B_{10} = B'_{00} B'_{11}$. But, the product of two right-handed quaternions is always another right handed quaternion. Similarily, the product of two left handed quaternions is another left handed quaternion. So, the L.H.S and R.H.S are only equal when both products are real numbers, i.e. $B_{01} B_{10} \in \mathbb{R}$ and $B'_{00} B'_{11} \in \mathbb{R}$. In this case, either both factors in the pair product are real numbers, or the versor of one factor is the quaternion conjugate of the versor of the other. Either way, the factors must commute, and $B'_{00} B'_{11}$ can be written $B'_{11} B'_{00}$, showing the divisor has same form, $A_{10} - A^{-1}_{01} A'_{11} A'_{00}$, that's on the other side. Similarly, we can show that the condition that the other divisor in the cross diagonal vanishes, is that the factors commute, and again the form must be equivalent to that on the other side. So, if any of the 4 divisors vanish in either left side inverse or right side inverse matrix, then the corresponding divisor vanishes also for the other matrix, and both left and right inverses either exist together or are non-existant together. Thus, *A matrix either has both left and right non-associative* $\times$ *product inverses, or neither.*

RESOLVING THE TWO-HAND QUATERNIONS:

Now, having reviewed the forms for these product inverses, and compared their various modifications, we need to return to the twisted product inverse and complete these expressions (32)-(33) and (43)-(44). These formulas make use of the two-hand quaternion representation, or `hexpe numbers`, and can be finally resolved and re-expressed in terms of just one-hand quaternions, i.e. the right-hand quaternions, again, by using the inverse formulas for `hexpe numbers`.

`Octonion Forms.` First, we shall use the rules of two-hand quaternion algebra to prove that numbers with octonion form have the same left and right inverses for the non-associative $\times$ product, despite the fact that the formulas in (49) show these quatro-quaternion inverses are otherwise generally differerent; and thus we shall resolve the inverse for these numbers to show their corresponding one hand, i.e. right hand, form.

Now, given a number with octonion form (5), we set, $A_{00} = B_{00} = A$, $A_{10} = B_{10} = B$, $A_{01} = B_{01} = -B^*$, and $A_{11} = B_{11} = A^*$. Substituting these into the formulas (49), we obtain,

$$\text{LEFT} \times \text{INVERSE } \hat{A}[..] = \qquad \qquad \text{RIGHT} \times \text{INVERSE } \hat{B}[..] =$$

$$\begin{pmatrix} \frac{\hat{1}}{\vdash (A' - (A^*)'^{-1} B(-B^*))} & \frac{\hat{1}}{\vdash (B - (-B^*)^{-1} A'(A^*)')} \\ \frac{\hat{1}}{\vdash ((-B^*)' - B'^{-1}(A^*)A)} & \frac{\hat{1}}{\vdash ((A^*) - A^{-1}(-B^*)'B')} \end{pmatrix} \quad | \quad \begin{pmatrix} \frac{\hat{1}}{\vdash (A - (A^*)^{-1}(-B^*)'B')} & \frac{\hat{1}}{\vdash (B - (-B^*)^{-1}(A^*)'A')} \\ \frac{\hat{1}}{\vdash ((-B^*)' - B'^{-1}A(A^*))} & \frac{\hat{1}}{\vdash ((A^*)' - A'^{-1}B(-B^*))} \end{pmatrix} \tag{50}$$

At first glance, these still look different, because of the dissimilar hand transforms appearing in the corresponding divisors. However, a careful inspection reveals that all the divisors are composed of one-hand quaternions only. This is because the products, $BB^* = B^*B = B'(B^*)' = (B^*)'B' = |B|^2$, etc.. i.e. the quaternions have the same square norm whether represented in their right hand or left hand format, and these norms are just real numbers. This means that each divisor is either completely a right-hand quaternion, or entirely left-handed.

Now we can use the commutation laws for pivots from two-hand quaternion algebra, which tells us that for any two right-hand quaternions, $X, Y \in \mathbb{H}_R$, we can write, $X' \cdot \hat{Y} = Y \cdot X$ and $X \cdot \hat{Y} = X \cdot Y$. This allows us to commute the divisors with the unit pivot parameter $\hat{1}$ appearing in the numerator, e.g.,

$$\frac{\hat{1}}{\vdash (A' - (A^*)'^{-1}B(-B^*))} \;=\; \frac{1}{(A' + (A^*)'^{-1}|B|^2)} \cdot \hat{1} \;=\; 1 \cdot \frac{1}{(A + (A^*)^{-1}|B|^2)} \tag{51}$$

Since the factor on the L.H.S is a left-hand quaternion, we can move it over to the R.H.S of the unit pivot, where it transforms into a right-hand quaternion, allowing us to remove the hat ˆ from the pivot $\hat{1}$, and represent that now as the ordinary unit 1.

When the factor on the L.H.S is already a right-hand quaternion, we can simply remove the hat ˆ from the pivot without further modification to the expression, e.g.,

$$\frac{\hat{1}}{\vdash (B - (-B^*)^{-1}A'(A^*)')} \;=\; \frac{1}{(B + (B^*)^{-1}|A|^2)} \cdot \hat{1} \;=\; \frac{1}{(B + (B^*)^{-1}|A|^2)} \cdot 1 \tag{52}$$

In this way, we can resolve the octonion form inverse into right-hand quaternion format. So, we have,

LEFT $\times$ INVERSE $A[..] =$ RIGHT $\times$ INVERSE $B[..] =$

$$\begin{pmatrix} \frac{1}{(A + (A^*)^{-1}|B|^2)} & \frac{1}{(B + (B^*)^{-1}|A|^2)} \\ \frac{1}{(-B^* - B^{-1}|A|^2)} & \frac{1}{(A^* + A^{-1}|B|^2)} \end{pmatrix} \quad | \quad \begin{pmatrix} \frac{1}{(A + (A^*)^{-1}B|^2)} & \frac{1}{(B + (B^*)^{-1}|A|^2)} \\ \frac{1}{(-B^* - B^{-1}|A|^2)} & \frac{1}{(A^* + A^{-1}|B|^2)} \end{pmatrix} \tag{53}$$

proving that they are identical. The unique inverse can then be re-expressed conveniently by

$$\begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix}_{\times}^{-1} = \begin{pmatrix} \frac{A^*}{|A|^2 + |B|^2} & \frac{B^*}{|B|^2 + |A|^2} \\ \frac{-B}{|B|^2 + |A|^2} & \frac{A}{|A|^2 + |B|^2} \end{pmatrix} \tag{54}$$

hence,

$$\begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix}_{\times}^{-1} = \frac{1}{D} \cdot \begin{pmatrix} A^* & B^* \\ -B & A \end{pmatrix} \tag{55}$$

where,

$$D = (A)(A^*) - (-B^*)(B) = |A|^2 + |B|^2$$

For the standard $\cdot$ product, the octonion form inverse can be obtained, for comparison, by substituting in (48), the corresponding (A, B) quaternions, we get,

$$\begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix}_{\cdot}^{-1} = \begin{pmatrix} \frac{1}{A + B^*(A^*)^{-1}B} & \frac{1}{B + A^*(B^*)^{-1}A} \\ \frac{1}{-B^* - AB^{-1}A^*} & \frac{1}{A^* + BA^{-1}B^*} \end{pmatrix} \tag{56}$$

from which we see that its associative inverse (56) is generally different from its non-associative inverse (53). Comparing terms, we find that the condition under which these two inverses are the same, is, $A \cdot B^* = B^* \cdot A$, which we can also write, $[A, B^*] = 0$. However, it is trivial to demonstrate that if, $[A, B^*] = 0$, then, $[A, B] = 0$, also, and visa versa. So, this means that the pair of quaternions (A, B) must commute, $AB = BA$, for the inverses to be identical. Looking ahead at equation (97), we see that this is the same condition required for the difference of squares to vanish, i.e. $o \times o - o \cdot o = 0$, so that the octonion form number then has the same square under the associative and non-associative products. Logically, reviewing the definitions (1) and (2), and recognising that they differ only by a simple *twisting*, we see immediately why commuting quaternions produce these same results. But, now we can make a stronger non-obvious statement, we deduce that, *if the parameters of an octonion form matrix non-commute, it must have a different associative inverse from its non-associative inverse.*

`General Forms.` Well, dealing with the octonion form is relatively straightforward. But, now we must examine the inverse formulas of the more general quatro-quaternion, and resolve its two-hand quaternion solutions into the right-hand. Examining the parameter solutions for the right (32), and left (43), non-associative inverses, reveals that all the divisors appear in the same basic form, $X' - Y$, or, $X - Y'$, $X, Y \in \mathbb{H}_R$, that is, the summation of a left-hand quaternion and right-hand quaternion. This is a particular type of simplified `hexpe number` that we refer to as a `bilateral factor` in our previous paper [PJ2] [2], because it has no cross terms from the product of left-hand bases with right-hand bases appearing in the sum. It therefore has a relatively simple `hexpe number` inverse formula which we reproduce below (from page 23 of that paper) for convenience:

BILATERAL FACTOR INVERSE:

$$h = h_0 \cdot 1 + h_{R1} \cdot i_R + h_{R2} \cdot j_R + h_{R3} \cdot k_R + h_{L1} \cdot i_L + h_{L2} \cdot j_L + h_{L3} \cdot k_L \qquad h, h^{-1} \in \mathbb{X}_n\ , \quad h_j, w_k \in \mathbb{R} \tag{57}$$

$$\begin{aligned}
h^{-1} &= (w_0 \cdot 1 \\
&+ w_{R1} \cdot i_R + w_{R2} \cdot j_R + w_{R3} \cdot k_R \\
&+ w_{L1} \cdot i_L + w_{L2} \cdot j_L + w_{L3} \cdot k_L \\
&+ w_{M1} \cdot i_M + w_{M2} \cdot j_M + w_{M3} \cdot k_M \\
&+ w_{A1} \cdot i_A + w_{A2} \cdot j_A + w_{A3} \cdot k_A \\
&+ w_{Z1} \cdot i_Z + w_{Z2} \cdot j_Z + w_{Z3} \cdot k_Z)/d
\end{aligned}
\qquad
\begin{aligned}
d = &\ (w_0 \cdot h_0 \\
&- w_{R1} \cdot h_{R1} - w_{R2} \cdot h_{R2} - w_{R3} \cdot h_{R3} \\
&- w_{L1} \cdot h_{L1} - w_{L2} \cdot h_{L2} - w_{L3} \cdot h_{L3}) \\
w_{M1} &= 2h_0 h_{R1} h_{L1} \\
w_{M2} &= 2h_0 h_{R2} h_{L2} \\
w_{M3} &= 2h_0 h_{R3} h_{L3}
\end{aligned} \tag{58}$$

$$\begin{aligned}
w_0 &= h_0(+h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2) & & \\
w_{R1} &= h_{R1}(-h_0^2 - h_{R1}^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2) & w_{A1} &= 2h_0 h_{R2} h_{L3} \\
w_{R2} &= h_{R2}(-h_0^2 - h_{R1}^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2) & w_{A2} &= 2h_0 h_{R3} h_{L1} \\
w_{R3} &= h_{R3}(-h_0^2 - h_{R1}^2 - h_{R2}^2 - h_{R3}^2 + h_{L1}^2 + h_{L2}^2 + h_{L3}^2) & w_{A3} &= 2h_0 h_{R1} h_{L2} \\
w_{L1} &= h_{L1}(-h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 - h_{L3}^2) & w_{Z1} &= 2h_0 h_{R3} h_{L2} \\
w_{L2} &= h_{L2}(-h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 - h_{L3}^2) & w_{Z2} &= 2h_0 h_{R1} h_{L3} \\
w_{L3} &= h_{L3}(-h_0^2 + h_{R1}^2 + h_{R2}^2 + h_{R3}^2 - h_{L1}^2 - h_{L2}^2 - h_{L3}^2) & w_{Z3} &= 2h_0 h_{R2} h_{L1}
\end{aligned}$$

In the above expressions, h, is our special `hexpe number` that is the sum of right-hand and left-hand quaternions. Then, h^{-1}, is the inverse of this number, given in terms of scalar weight factors, the w's, a scalar normalizing determinant divisor, d, the 16-dimensional basis units for the `hexpe number`: the seven, $1, i_R, j_R, k_R, i_L, j_L, k_L$, from the original bilateral factor, and an additional nine, $i_M, j_M, k_M, i_A, j_A, k_A, i_Z, j_Z, k_Z$, required to complete the inverse. Notice that the weight factors, w's, all vanish for the M-A-Z units, whenever the scalar, h_0, is zero. When this scalar is zero our bilateral factor is the sum of a right pure quaternion and left pure quaternion. In this case, the inverse, h^{-1}, is also a pure right left quaternion, and the inverse of a bilateral factor is just another bilateral factor. Otherwise, the inverse is a general `hexpe number` with typically all 16-dimensions present. We can simplify this expression (58), for the inverse, by introducing a few additional symbolic definitions. For any given `hexpe number`, h, let, $R(h)$, be the right pure quaternion component, and, $L(h)$, be the left pure quaternion component. Then, let's add right conjugation and left conjugation operators, so that, while, h^* is the conjugate of both right and left basis units simultaneously, h^{*_R} is the conjugate of the right basis only, and, h^{*_L} is the conjugate of the left basis only. Then, we replace all the ijk units for the M-A-Z in h^{-1} with their equivalent R-L pair products, $i_M = i_R i_L, \ldots, j_A = k_R i_L, \ldots$ etc., Our `bilateral factor` inverse formula becomes,

$$h^{-1} = \frac{h_0^2 h^* + r^2 h^{*_R} + l^2 h^{*_L} + 2h_0 R(h) L(h)}{(h_0^2 + r^2 + l^2)^2 - 4r^2 l^2} \qquad h, h^{-1} \in \mathbb{X}_n\ , \quad R(h) \in \mathbb{H}_R\ , \quad L(h) \in \mathbb{H}_L \tag{59}$$

`where,`

$$\begin{aligned}
h &= h_0 + R(h) + L(h) & d &= h_0^4 + 2h_0^2 r^2 + 2h_0^2 l^2 + r^4 - 2r^2 l^2 + l^4 \\
h^* &= h_0 - R(h) - L(h) & &= (h_0^2 + r^2 + l^2)^2 - 4r^2 l^2 \\
h^{*_R} &= h_0 - R(h) + L(h) & &= (h_0^2 + (r + l)^2)(h_0^2 + (r - l)^2) \\
h^{*_L} &= h_0 + R(h) - L(h) & & \\
R(h) &= h_{R1} \cdot i_R + h_{R2} \cdot j_R + h_{R3} \cdot k_R & & \\
L(h) &= h_{L1} \cdot i_L + h_{L2} \cdot j_L + h_{L3} \cdot k_L & & \\
r^2 &= R(h) R(h)^* = |R(h)|^2 = h_{R1}^2 + h_{R2}^2 + h_{R3}^2 & & \\
l^2 &= L(h) L(h)^* = |L(h)|^2 = h_{L1}^2 + h_{L2}^2 + h_{L3}^2 & &
\end{aligned}$$

Another way to express this same formula (59) is,

$$h^{-1} = (h_0^2(h_0 - R(h) - L(h)) + r^2(h_0 - R(h) + L(h)) + l^2(h_0 + R(h) - L(h)) + 2h_0 R(h)L(h))/d \tag{60}$$

Then, when, $h = X + Y' : \ X, Y \in \mathbb{H}_R$, we have, $h_0 = X_0 + Y_0$, $R(h) = X - X_0$, $L(h) = Y' - Y_0$. So, we can re-write this inverse,

$$h^{-1} = (\ h_0^2(X^* + Y'^*) + r^2(X^* + Y') + l^2(X + Y'^*) + 2h_0(X - X_0)(Y' - Y_0)\ )/d \tag{61}$$

We now have the inverse expressed in terms of R-H and L-H quaternions appearing only in the numerator, the denominator having a pure scalar factor, d. This allows us to invert the divisors in the right (32), and left (43), *twisted product* inverse expressions. All the formulas we have to resolve there have the form, $h^{-1} \cdot Z \cdot \hat{1}$, or $h^{-1} \cdot Z' \cdot \hat{1}$, where, $Z \in \mathbb{H}_R$, and, $h = X + Y'$, with, $X, Y \in \mathbb{H}_R$. For the the first form, we apply the associative and distributive laws from the two-hand quaternion algebra, to obtain (62) and (63),

$$h^{-1} \cdot Z \cdot \hat{1} = (h^{-1} \cdot Z) \cdot \hat{1} \tag{62}$$

$$= (\ h_0^2((X^*Z) \cdot \hat{1} + (Y'^*Z) \cdot \hat{1}) + r^2((X^*Z) \cdot \hat{1} + (Y'Z) \cdot \hat{1}) + l^2((XZ) \cdot \hat{1} + (Y'^*Z) \cdot \hat{1}) + 2h_0(X - X_0)((Y' - Y_0)Z) \cdot \hat{1}\ )/d \tag{63}$$

$$= (\ h_0^2((X^*Z) \cdot \hat{1} + Z(Y'^* \cdot \hat{1})) + r^2((X^*Z) \cdot \hat{1} + Z(Y' \cdot \hat{1})) + l^2((XZ) \cdot \hat{1} + Z(Y'^* \cdot \hat{1})) + 2h_0(X - X_0)Z((Y' - Y_0) \cdot \hat{1})\ )/d \tag{64}$$

$$= (\ h_0^2((X^*Z) \cdot 1 + Z(1 \cdot Y^*)) + r^2((X^*Z) \cdot 1 + Z(1 \cdot Y) + l^2((XZ) \cdot 1 + Z(1 \cdot Y^*)) + 2h_0(X - X_0)Z(1 \cdot (Y - Y_0))\ )/d \tag{65}$$

$$= (\ h_0^2(X^*Z + ZY^*) + r^2(X^*Z + ZY) + l^2(XZ + ZY^*) + 2h_0(X - X_0)Z(Y - Y_0)\ )/d \tag{66}$$

Using the fact that L-H quaternions commute with R-H quaternions, we permute all the L-H quaternions to the R-H-S of the R-H quaternions; the associative law for pivots re-groups these L-H quaternions with pivots (64). Then, using the commutative law for pivots we move each L-H quaternion over to the R-H-S of the unit $\hat{1}$ pivot, where it changes into a R-H quaternion, and the unit pivot becomes the ordinary scalar unit 1, so we remove the hat $\hat{}$, giving us the final right-hand form (65), which we can simplify removing parentheses etc..to obtain (66); and we proceed similarily for the second form,

$$h^{-1} \cdot Z' \cdot \hat{1} = (h^{-1} \cdot Z') \cdot \hat{1} \tag{67}$$

$$= (\ h_0^2((X^*Z') \cdot \hat{1} + (Y'^*Z') \cdot \hat{1}) + r^2((X^*Z') \cdot \hat{1} + (Y'Z') \cdot \hat{1}) + l^2((XZ') \cdot \hat{1} + (Y'^*Z') \cdot \hat{1}) + 2h_0(X - X_0)((Y' - Y_0)Z') \cdot \hat{1}\ )/d \tag{68}$$

$$= (\ h_0^2(X^*(Z' \cdot \hat{1}) + (Y'^*Z') \cdot \hat{1}) + r^2(X^*(Z' \cdot \hat{1}) + (Y'Z') \cdot \hat{1}) + l^2(X(Z' \cdot \hat{1}) + (Y'^*Z') \cdot \hat{1}) + 2h_0(X - X_0)(((Y' - Y_0)Z') \cdot \hat{1})\ )/d \tag{69}$$

$$= (\ h_0^2(X^*(1 \cdot Z) + 1 \cdot (ZY^*)) + r^2(X^*(1 \cdot Z) + 1 \cdot (ZY)) + l^2(X(1 \cdot Z) + 1 \cdot (ZY^*)) + 2h_0(X - X_0)(1 \cdot (Z(Y - Y_0)))\ )/d \tag{70}$$

$$= (\ h_0^2(X^*Z + ZY^*) + r^2(X^*Z + ZY) + l^2(XZ + ZY^*) + 2h_0(X - X_0)Z(Y - Y_0)\ )/d \tag{71}$$

However, we now make use of the hand transformation product rule, $(AB)' = B'A'$, which reverses the factors, to resolve the expressions, e.g. $(Y'^*Z') \cdot \hat{1} = 1 \cdot (Y'^*Z')' = 1 \cdot (ZY^*) = ZY^*$. We must always remember to permute the entire L-H quaternion to the R-H-S of the pivot in a single move, and if this left-hand quaternion is composed of the products of many other L-H quaternion factors, then, after the move, we use the hand transformation rule to reverse the order of the factors in converting them to the R-H. We then obtain the one-hand form (71), where everything is in the right-hand quaternion representation.

Notice that (66) and (71) are the same. That is, $h^{-1} \cdot Z' \cdot \hat{1} = h^{-1} \cdot Z \cdot \hat{1}$, and it doesn't matter whether the Z parameter in this expression is in left-hand or right-hand format. This is a unique feature of this particular product expression, because of the appearence of the unit pivot $\hat{1}$. If there were any other non-trivial quaternion there (i.e. not proportional to the unit scalar), we'd usually obtain two different results, e.g. $h^{-1} \cdot Z' \cdot \hat{Q}$ and $h^{-1} \cdot Z \cdot \hat{Q}$ would not generally be the same. If we replace the unit pivot $\hat{1}$, with a general pivot $\hat{Q}$, in (62) and (67), to compare, we'd find that terms that end up with $Z \cdot Q$ in their expressions, in the first case, produce $Q \cdot Z$ instead, in the second. Of course, when $Q = 1$, this difference vanishes. Or, more generally, if $[Q, Z] = 0$, the results are again the same. A consequence of this is that it doesn't matter whether the numerators in the right (32), or left (43), inverse formulas, are in R-H or L-H format, we'd obtain the same results. So, we can just remove all the prime marks $'$ from these numerators.

The four divisors in the twisted product right (32) and left (43) inverse formulas, then, determine the difference between these right and left inverses. Therefore, we now start with the formula,

$$\texttt{Given,}\quad X,Y,Z\in\mathbb{H}_R;\;\; h=X+Y'\in\mathbb{X}_n,\;\; \mathcal{S}X=X_0,\;\mathcal{S}Y=Y_0,\;\mathcal{V}X=X-X_0,\;\mathcal{V}Y=Y-Y_0,\;\texttt{etc..}$$

$$h^{-1}\cdot Z\cdot\hat{1}= \tag{72}$$

$$\frac{(\,(X_0+Y_0)^2(X^*Z+ZY^*)+|X-X_0|^2(X^*Z+ZY)+|Y-Y_0|^2(XZ+ZY^*)+2(X_0+Y_0)(X-X_0)Z(Y-Y_0)\,)}{((X_0+Y_0)^2+|X-X_0|^2+|Y-Y_0|^2)^2-4|X-X_0|^2|Y-Y_0|^2}$$

$$=\frac{(\,(\mathcal{S}(X+Y))^2(X^*Z+ZY^*)+|\mathcal{V}X|^2(X^*Z+ZY)+|\mathcal{V}Y|^2(XZ+ZY^*)+2(\mathcal{S}(X+Y))(\mathcal{V}X)Z(\mathcal{V}Y)\,)}{((\mathcal{S}(X+Y))^2+|\mathcal{V}X|^2+|\mathcal{V}Y|^2)^2-4|\mathcal{V}X|^2|\mathcal{V}Y|^2} \tag{73}$$

and replace the X,Y,Z parameters with the corresponding quaternions from our twisted product inverse expressions to obtain the right-hand representation of these results. In (72) we use the modern notation, but sometimes it helps to see things Hamilton's way. So, we've borrowed some of W.R. Hamilton's original notation, his scalar and vector operators, $S(\cdot)$ and $V(\cdot)$, which extract the appropriate parts of the quaternion, and we illustrate the same formula written alternatively in (73) using the older notation. Hamilton introduced six operators, S,V,K,N,T,U, for the, scalar, vector, conjugate, norm, tensor, and versor, parts of a quaternion. We've kept the modern conjugate * and norm $|\cdot|^2$, however, since we think today that makes things clearer. With the prime $'$ marks removed from the numerators,

`right` $\times$ `inverse` (32):

$$\hat{B}_{00} = (A_{11}A_{00}-A'_{01}A'_{10})^{-1}\cdot A_{11}\cdot\hat{1}\,,\qquad X=A_{11}A_{00}\,,\;Y=-A_{10}A_{01}\,,\;Z=A_{11} \tag{74}$$
$$\hat{B}_{10} = (A'_{10}A'_{01}-A_{00}A_{11})^{-1}\cdot A_{10}\cdot\hat{1}\,,\qquad X=-A_{00}A_{11}\,,\;Y=A_{01}A_{10}\,,\;Z=A_{10} \tag{75}$$
$$\hat{B}_{01} = (A_{01}A_{10}-A'_{11}A'_{00})^{-1}\cdot A_{01}\cdot\hat{1}\,,\qquad X=A_{01}A_{10}\,,\;Y=-A_{00}A_{11}\,,\;Z=A_{01} \tag{76}$$
$$\hat{B}_{11} = (A'_{00}A'_{11}-A_{10}A_{01})^{-1}\cdot A_{00}\cdot\hat{1}\,,\qquad X=-A_{10}A_{01}\,,\;Y=A_{11}A_{00}\,,\;Z=A_{00} \tag{77}$$

`left` $\times$ `inverse` (43):

$$\hat{A}_{00} = (B'_{11}B'_{00}-B_{10}B_{01})^{-1}\cdot B_{11}\cdot\hat{1}\,,\qquad X=-B_{10}B_{01}\,,\;Y=B_{00}B_{11}\,,\;Z=B_{11} \tag{78}$$
$$\hat{A}_{10} = (B'_{10}B'_{01}-B_{11}B_{00})^{-1}\cdot B_{10}\cdot\hat{1}\,,\qquad X=-B_{11}B_{00}\,,\;Y=B_{01}B_{10}\,,\;Z=B_{10} \tag{79}$$
$$\hat{A}_{01} = (B_{01}B_{10}-B'_{00}B'_{11})^{-1}\cdot B_{01}\cdot\hat{1}\,,\qquad X=B_{01}B_{10}\,,\;Y=-B_{11}B_{00}\,,\;Z=B_{01} \tag{80}$$
$$\hat{A}_{11} = (B_{00}B_{11}-B'_{01}B'_{10})^{-1}\cdot B_{00}\cdot\hat{1}\,,\qquad X=B_{00}B_{11}\,,\;Y=-B_{10}B_{01}\,,\;Z=B_{00} \tag{81}$$

these expressions then yield the non-associative inverses. Note that we reverse the order of the factors when identifying the Y parameter, i.e. $Y'=-A'_{01}A'_{10}\implies Y=(Y')'=(-A'_{01}A'_{10})'=-A_{10}A_{01}$, etc. since the hand transformation rule for products applies. Also, notice that if we exchange the X and Y parameters in (73) the numerator changes, but the denominator remains unchanged. This means that (74) and (77) now have the same denominator, for example, and all of our (73)-type denominators fall into such pairs. Looking back at the original right (33) and left (44) matrix inverse formulas, we see that these same equal pairs come from those previous denominators that were hand transforms of each other, where we found, $D_4=D'_1$ and $D_3=D'_2$.

Three twisted expressions of the form, $XZ+ZY$, and one vector product form, $(\mathcal{V}X)Z(\mathcal{V}Y)$, appear in (73). These are the only terms that result in general quaternions, the other component factors and parameters are all effectively scalars. Although quaternions don't usually commute, in the particular assignments to the three X,Y,Z, parameters above, we can effectively permute this inside Z parameter to the outside of these twisted expressions. We need only one extra symbolic device to facilitate recognition of this commuting property. Note, the variables, X, Y, are both themselves formed from products of two quaternion factors. In order to commute the Z with these, we have to reverse the order of the factors within the X or Y, so let us define a symbolic operator to denote this: if $X=PQ$ then $\overleftarrow{X}=QP$.

Consider, for example, the evaluation of the B_{00} in (74). When we substitute values for the X,Y,Z, in (X^*Z+ZY^*), we obtain, $((A_{11}A_{00})^*A_{11}+A_{11}(-A_{10}A_{01})^*)$. But, conjugation reverses the order of products, so, $(A_{11}A_{00})^*A_{11}=A^*_{00}A^*_{11}A_{11}=A^*_{00}|A_{11}|^2=|A_{11}|^2A^*_{00}=A_{11}A^*_{11}A^*_{00}$. Therefore, we can write, $(X^*Z+ZY^*)=Z(\overleftarrow{X}^*+Y^*)$, and we have effectively commuted the inside parameter to the L.H.S of the twisted expression, which allows us to factor this out on the left. This peculiar commuting property is a special case situation that arises from the particular selection the four quaternions, $(A_{00}, A_{10}, A_{01}, A_{11})$, when allocated to the three parameters, X, Y, Z, and is a unique feature of our matrix inverse formulas. The reversing operation, $\overleftarrow{X}$, helps to un-twist and simplify these expressions. Now, consider the vector product, $(VX)Z(VY)$. On substitution, $(V(A_{11}A_{00}))A_{11}(V(-A_{10}A_{01}))$. On the left we have, $(V(A_{11}A_{00}))A_{11} = (A_{11}A_{00} - S(A_{11}A_{00}))A_{11} = (A_{11}A_{00}A_{11} - A_{11}S(A_{00}A_{11}))$. So, again we can effectively commute the variables and re-write the expression with Z moving to the left, $(VX)Z(VY) = Z(V\overleftarrow{X})(VY)$.

`For the right × inverse:`

$$
\begin{array}{ll}
B_{00}:\ X = A_{11}A_{00}\ ,\ Y = -A_{10}A_{01}\ ,\ Z = A_{11} & B_{01}:\ X = A_{01}A_{10}\ ,\ Y = -A_{00}A_{11}\ ,\ Z = A_{01} \\
 & \\
X^*Z + ZY^* \ = A_{11}(A_{11}^*A_{00}^* - A_{01}^*A_{10}^*) = Z(\overleftarrow{X}^* + Y^*) & X^*Z + ZY^* \ = -A_{01}(A_{11}^*A_{00}^* - A_{01}^*A_{10}^*) = Z(\overleftarrow{X}^* + Y^*) \\
X^*Z + ZY \ = A_{11}(A_{11}^*A_{00}^* - A_{10}A_{01}) = Z(\overleftarrow{X}^* + Y) & X^*Z + ZY \ = -A_{01}(A_{00}A_{11} - A_{01}^*A_{10}^*) = Z(\overleftarrow{X}^* + Y) \\
XZ \ + ZY^* \ = A_{11}(A_{00}A_{11} - A_{01}^*A_{10}^*) = Z(\overleftarrow{X} \ + Y^*) & XZ \ + ZY^* \ = -A_{01}(A_{11}^*A_{00}^* - A_{10}A_{01}) = Z(\overleftarrow{X} \ + Y^*) \\
(VX)Z(VY) = A_{11}(V(A_{00}A_{11}))(V(-A_{10}A_{01})) = Z(V\overleftarrow{X})(VY) & (VX)Z(VY) = -A_{01}(V(A_{10}A_{01}))(V(A_{00}A_{11})) = Z(V\overleftarrow{X})(VY) \\
 & \\
B_{10}:\ X = -A_{00}A_{11}\ ,\ Y = A_{01}A_{10}\ ,\ Z = A_{10} & B_{11}:\ X = -A_{10}A_{01}\ ,\ Y = A_{11}A_{00}\ ,\ Z = A_{00} \\
 & \\
X^*Z + ZY^* \ = -(A_{11}^*A_{00}^* - A_{01}^*A_{10}^*)A_{10} = (X^* + \overleftarrow{Y}^*)Z & X^*Z + ZY^* \ = (A_{11}^*A_{00}^* - A_{01}^*A_{10}^*)A_{00} = (X^* + \overleftarrow{Y}^*)Z \\
X^*Z + ZY \ = -(A_{11}^*A_{00}^* - A_{10}A_{01})A_{10} = (X^* + \overleftarrow{Y})Z & X^*Z + ZY \ = (A_{00}A_{11} - A_{01}^*A_{10}^*)A_{00} = (X^* + \overleftarrow{Y})Z \\
XZ \ + ZY^* \ = -(A_{00}A_{11} - A_{01}^*A_{10}^*)A_{10} = (X \ + \overleftarrow{Y}^*)Z & XZ \ + ZY^* \ = (A_{11}^*A_{00}^* - A_{10}A_{01})A_{00} = (X \ + \overleftarrow{Y}^*)Z \\
(VX)Z(VY) = -(V(A_{00}A_{11}))(V(A_{10}A_{01}))A_{10} = (VX)(V\overleftarrow{Y})Z & (VX)Z(VY) = (V(-A_{10}A_{01}))(V(A_{00}A_{11}))A_{00} = (VX)(V\overleftarrow{Y})Z
\end{array}
\tag{82}
$$

The un-twisting of all the expressions for the inverse matrix components reveals a simpler pattern. The three conjugated variations of the twisted form, $XZ + ZY$, each split into two, $Z(\overleftarrow{X} + Y)$ and $(X + \overleftarrow{Y})Z$, where Z factors out on the left and the right. For the right $\times$ inverse, Z moves left in the top row, but right in the bottom row. While, for the left $\times$ inverse, Z moves right in the first column, and left in the second column. By extracting a minus sign to the outside, for the cross terms, e.g. $\{B_{10}, B_{01}\}$ above, we can then re-write our matrix formulas in terms of the usual four quaternion numerators, i.e. $(A_{11}, -A_{10}, -A_{01}, A_{00})$, that appear in our other matrix inverse formulas.

`For the left × inverse:`

$$
\begin{array}{ll}
A_{00}:\ X = -B_{10}B_{01}\ ,\ Y = B_{00}B_{11}\ ,\ Z = B_{11} & A_{01}:\ X = B_{01}B_{10}\ ,\ Y = -B_{11}B_{00}\ ,\ Z = B_{01} \\
 & \\
X^*Z + ZY^* \ = (B_{00}^*B_{11}^* - B_{01}^*B_{10}^*)B_{11} = (X^* + \overleftarrow{Y}^*)Z & X^*Z + ZY^* \ = -B_{01}(B_{00}^*B_{11}^* - B_{01}^*B_{10}^*) = Z(\overleftarrow{X}^* + Y^*) \\
X^*Z + ZY \ = (B_{11}B_{00} - B_{01}^*B_{10}^*)B_{11} = (X^* + \overleftarrow{Y})Z & X^*Z + ZY \ = -B_{01}(B_{11}B_{00} - B_{01}^*B_{10}^*) = Z(\overleftarrow{X}^* + Y) \\
XZ \ + ZY^* \ = (B_{00}^*B_{11}^* - B_{10}B_{01})B_{11} = (X \ + \overleftarrow{Y}^*)Z & XZ \ + ZY^* \ = -B_{01}(B_{00}^*B_{11}^* - B_{10}B_{01}) = Z(\overleftarrow{X} \ + Y^*) \\
(VX)Z(VY) = (V(-B_{10}B_{01}))(V(B_{11}B_{00}))B_{11} = (VX)(V\overleftarrow{Y})Z & (VX)Z(VY) = -B_{01}(V(B_{10}B_{01}))(V(B_{11}B_{00})) = Z(V\overleftarrow{X})(VY) \\
 & \\
A_{10}:\ X = -B_{11}B_{00}\ ,\ Y = B_{01}B_{10}\ ,\ Z = B_{10} & A_{11}:\ X = B_{00}B_{11}\ ,\ Y = -B_{10}B_{01}\ ,\ Z = B_{00} \\
 & \\
X^*Z + ZY^* \ = -(B_{00}^*B_{11}^* - B_{01}^*B_{10}^*)B_{10} = (X^* + \overleftarrow{Y}^*)Z & X^*Z + ZY^* \ = B_{00}(B_{00}^*B_{11}^* - B_{01}^*B_{10}^*) = Z(\overleftarrow{X}^* + Y^*) \\
X^*Z + ZY \ = -(B_{00}^*B_{11}^* - B_{10}B_{01})B_{10} = (X^* + \overleftarrow{Y})Z & X^*Z + ZY \ = B_{00}(B_{00}^*B_{11}^* - B_{10}B_{01}) = Z(\overleftarrow{X}^* + Y) \\
XZ \ + ZY^* \ = -(B_{11}B_{00} - B_{01}^*B_{10}^*)B_{10} = (X \ + \overleftarrow{Y}^*)Z & XZ \ + ZY^* \ = B_{00}(B_{11}B_{00} - B_{01}^*B_{10}^*) = Z(\overleftarrow{X} \ + Y^*) \\
(VX)Z(VY) = -(V(B_{11}B_{00}))(V(B_{10}B_{01}))B_{10} = (VX)(V\overleftarrow{Y})Z & (VX)Z(VY) = B_{00}(V(B_{11}B_{00}))(V(-B_{10}B_{01})) = Z(V\overleftarrow{X})(VY)
\end{array}
\tag{83}
$$

To see this, first notice that we may now factor out the Z parameter from the entire expression in (73), and we can thus re-write this formula (73) in two ways, $Z \cdot \rho_X(X,Y)$ and $\rho_Y(X,Y) \cdot Z$;

$$
Z \cdot \rho_X(X,Y) \ = \ Z \cdot \frac{(\,(S(X+Y))^2(\overleftarrow{X}^* + Y^*) + |VX|^2(\overleftarrow{X}^* + Y) + |VY|^2(\overleftarrow{X} + Y^*) + 2(S(X+Y))(V\overleftarrow{X})(VY)\,)}{((S(X+Y))^2 + |VX|^2 + |VY|^2)^2 - 4|VX|^2|VY|^2} \tag{84}
$$

$$
\rho_Y(X,Y) \cdot Z \ = \ \frac{(\,(S(X+Y))^2(X^* + \overleftarrow{Y}^*) + |VX|^2(X^* + \overleftarrow{Y}) + |VY|^2(X + \overleftarrow{Y}^*) + 2(S(X+Y))(VX)(V\overleftarrow{Y})\,)}{((S(X+Y))^2 + |VX|^2 + |VY|^2)^2 - 4|VX|^2|VY|^2} \cdot Z \tag{85}
$$

The non-associative $\times$ product inverse formulas can then be written in the usual matrix form,

RIGHT $\times$ INVERSE (33):

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{A_{11}}{D_1 \dashv} & \dfrac{-A_{01}}{D_3 \dashv} \\ \\ \dfrac{-A_{10}}{\vdash D_2} & \dfrac{A_{00}}{\vdash D_4} \end{pmatrix} = \begin{pmatrix} A_{11} \cdot D_1^{-1} & -A_{01} \cdot D_3^{-1} \\ \\ D_2^{-1} \cdot (-A_{10}) & D_4^{-1} \cdot A_{00} \end{pmatrix} \tag{86}$$

`where,`

$$D_1^{-1} = \rho_X(A_{11}A_{00}, -A_{10}A_{01}), \;\; D_2^{-1} = -\rho_Y(-A_{00}A_{11}, A_{01}A_{10}),$$
$$D_4^{-1} = \rho_Y(-A_{10}A_{01}, A_{11}A_{00}), \;\; D_3^{-1} = -\rho_X(A_{01}A_{10}, -A_{00}A_{11}).$$

where everything is once more expressed in right-hand quaternions. Right $\times$ inverse one-hand format (86) is the equivalent of its two-hand format (33), and left $\times$ inverse one-hand format (87) the equivalent of two-hand (44). The formulas (84) and (85) do not reflect the minus sign extractions for the matrix cross terms, as exhibited in the expanded expressions, for $\{B_{10}, B_{01}\}$ in (82) and $\{A_{10}, A_{01}\}$ in (83), so we include these minus signs in the divisor definitions here for the D_μ, $\mu = 1, 2, 3, 4$. Notice also, that while all our divisors were on the left, in the two-hand format, here we have a mixing of left and right divisions within the same matrix once we've converted to the one-hand format. The divisors move to the right in the top row, and to the left in bottom row, for the right $\times$ inverse.

LEFT $\times$ INVERSE (44):

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} \dfrac{B_{11}}{\vdash D_1} & \dfrac{-B_{01}}{D_3 \dashv} \\ \\ \dfrac{-B_{10}}{\vdash D_2} & \dfrac{B_{00}}{D_4 \dashv} \end{pmatrix} = \begin{pmatrix} D_1^{-1} \cdot B_{11} & -B_{01} \cdot D_3^{-1} \\ \\ D_2^{-1} \cdot (-B_{10}) & B_{00} \cdot D_4^{-1} \end{pmatrix} \tag{87}$$

`where,`

$$D_1^{-1} = \rho_Y(-B_{10}B_{01}, B_{00}B_{11}), \;\; D_2^{-1} = -\rho_Y(-B_{11}B_{00}, B_{01}B_{10}),$$
$$D_4^{-1} = \rho_X(B_{00}B_{11}, -B_{10}B_{01}), \;\; D_3^{-1} = -\rho_X(B_{01}B_{10}, -B_{11}B_{00}).$$

While, the divisors move to the left in the first column, and to the right in the second column, for the left $\times$ inverse. Observe too, that if we swap the X and Y parameters in the $\rho_X(X,Y)$ of (84), we'd almost get the expression for $\rho_Y(X,Y)$ in (85), except for the last vector product term $(VX)(V\overleftarrow{Y})$ in the numerator. If $(V\overleftarrow{Y})(VX) = (VX)(V\overleftarrow{Y})$, then we could write, $\rho_Y(X,Y) = \rho_X(Y,X)$, and our four divisors, D_μ, $\mu = 1, 2, 3, 4$, would reduce to just two. In fact, we'd get, $D_1 = D_4$ and $D_2 = D_3$, for both right and left $\times$ inverses. Looking back at the two-hand formulas, (33) and (44), we see that these are just the very same divisor pairs that were previously hand-transforms of each other.

But, this exchange symmetry is broken by the presence of the vector product term (by vector product we mean the product involving quaternion vectors, not to be confused with the cross product of vector alegbra). That hand transform distinction has become reflected in presence of the symmetry breaking vector product term that shows up in the one-hand representation, effectively manifesting *a part* of that characteristic which was previously encapsulated in the two-hand format by the appearence of alternating hands among the divisors—the *other part* of the characteristic is now manifest in the mixing of left side and right side divisions within the matrix.

The information, $D_4 = D_1'$ and $D_3 = D_2'$, previously captured by that check mark $'$, is now described by the distinction between $(V\overleftarrow{Y})(VX)$ and $(VX)(V\overleftarrow{Y})$, and *relative side swapping* (i.e. $D_1^{-1} \cdot A_{11} \mapsto A_{11} \cdot D_1^{-1}$ while $D_4^{-1} \cdot A_{00} \mapsto D_4^{-1} \cdot A_{00}$), among the divisor pairs, so that when these vector factors commute, i.e. $[VX, V\overleftarrow{Y}] = 0$, the divisors fall into pairs of equal *values*, and the two-hand alternating difference is only then reflected in the *permutation* order of the factors exhibited by this particular swapping of left division with right division for one (but not both) member of each divisor pair. The differences, $D_4^{-1} - D_1^{-1}$, and $D_3^{-1} - D_2^{-1}$, in (86, 87), are given by the form;

$$\rho_Y(X,Y) - \rho_X(Y,X) = \frac{2(\mathcal{S}(X+Y))[(VX)(V\overleftarrow{Y}) - (V\overleftarrow{Y})(VX)]}{((\mathcal{S}(X+Y))^2 + |VX|^2 + |VY|^2)^2 - 4|VX|^2|VY|^2} \tag{88}$$

But, although, generally, $D_4 \neq D_1$, $D_3 \neq D_2$, here, a more thorough inspection reveals, $D_1 = D_2$, $D_3 = D_4$, for the right $\times$ inverse, and, $D_1 = D_3, D_2 = D_4$, for the left $\times$ inverse. The difference in the main diagonal divisors is the same as that difference in the cross diagonal divisors, i.e. $(D_4^{-1} - D_1^{-1}) = \pm(D_3^{-1} - D_2^{-1})$. And when $[VX, V\overleftarrow{Y}] = 0$, so that the expression (88) vanishes, all four divisors have the same value, $D_1 = D_2 = D_3 = D_4$. Let's see why.

Now, we can demonstrate that, $S(\overleftarrow{X}) = S(X)$, $|V(\overleftarrow{X})|^2 = |V(X)|^2$, $\forall\ X = PQ,\ P,Q \in \mathbb{H}_R$, and, $S(X+Y) = S(X)+S(Y)$. So, we may replace the two scalars, $S(X+Y)$ and $|V(X)|^2$, in (84), by the equivalents, $S(\overleftarrow{X}+Y)$ and $|V(\overleftarrow{X})|^2$, respectively, and similarily, replace, $S(X+Y)$ and $|V(Y)|^2$, in (85), by, $S(X+\overleftarrow{Y})$ and $|V(\overleftarrow{Y})|^2$. Then we can replace the two formula expressions with a single formula, and write, $\rho_X(X,Y) = \rho(\overleftarrow{X},Y)$, and $\rho_Y(X,Y) = \rho(X,\overleftarrow{Y},)$. Where,

$$\rho(X,Y) = \frac{(\,(S(X+Y))^2(X^*+Y^*) + |V(X)|^2(X^*+Y) + |V(Y)|^2(X+Y^*) + 2(S(X+Y))(V(X))(V(Y))}{((S(X+Y))^2 + |V(X)|^2 + |V(Y)|^2)^2 - 4|V(X)|^2|V(Y)|^2} \tag{89}$$

Note that, $\rho(Y,X) \neq \rho(X,Y)$, because of the vector product term, $V(X)V(Y)$, in the numerator. However, observe that, $\rho(-X,-Y) = -\rho(X,Y)$, so the non-associative $\times$ product inverse can be simplified to have just two divisors,

RIGHT $\times$ INVERSE $(33;86)$:

$$\begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} A_{11}\cdot D_1^{-1} & -A_{01}\cdot D_2^{-1} \\ D_1^{-1}\cdot(-A_{10}) & D_2^{-1}\cdot A_{00} \end{pmatrix} = \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} \times \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} \tag{90}$$

`where,`

$D_1^{-1} = \rho(A_{00}A_{11}, -A_{10}A_{01}), \qquad D_2^{-1} = \rho(-A_{10}A_{01}, A_{00}A_{11}).$

The divisors can then be factored out and the formula represented by a twisted product multiplication between the adjoint of the original matrix and a diagonal matrix of divisor values. For the right $\times$ inverse the diagonal divisor matrix in (90) appears on the right, and can be compared, for example, with the similar diagonal divisor matrix extraction for the right $\cdot$ inverse given previously in (40).

LEFT $\times$ INVERSE $(44;87)$:

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} = \begin{pmatrix} D_1^{-1}\cdot B_{11} & -B_{01}\cdot D_1^{-1} \\ D_2^{-1}\cdot(-B_{10}) & B_{00}\cdot D_2^{-1} \end{pmatrix} = \begin{pmatrix} D_1^{-1} & 0 \\ 0 & D_2^{-1} \end{pmatrix} \times \begin{pmatrix} B_{11} & -B_{01} \\ -B_{10} & B_{00} \end{pmatrix} \tag{91}$$

`where,`

$D_1^{-1} = \rho(-B_{10}B_{01}, B_{11}B_{00}), \qquad D_2^{-1} = \rho(B_{11}B_{00}, -B_{10}B_{01}).$

For the left $\times$ inverse, the diagonal divisor matrix in (91) appears on the left, which can be compared again to the left associative $\cdot$ inverse formula (46). These right and left side non-associative inverses can be verified by taking the $\times$ product with the original matrix. In this case, expressions of the form on the L.H.S of (92) appear in the main diagonal, while the cross diagonal terms easily vanish, and it can be shown from (89) that the equation (92) holds for general X, Y, right hand quaternions, so the inverses can be readily confirmed.

$$X\cdot\rho(X,Y) + \rho(X,Y)\cdot Y \;=\; 1 \tag{92}$$

$$X\cdot\hat{\rho} + Y'\cdot\hat{\rho} \;=\; \hat{1} \tag{93}$$

$$(X+Y')\cdot\hat{\rho} \;=\; \hat{1} \tag{94}$$

$$\hat{\rho} \;=\; \frac{1}{X+Y'}\cdot\hat{1} = \frac{\hat{1}}{\vdash (X+Y')} \tag{95}$$

If we start with this equation, (92), and apply our two-hand quaternion techniques, we can un-twist the expression, and recover the equivalent `bilateral factor` two-hand form, $(X+Y')^{-1}$, which we started out with, once again.

THE PROPERTIES OF THE QUATRO-QUATERNIONS:

A number of basic properties of the algebra of these quatro-quaternions can be easily demonstrated. The derivation of the non-associative inverse, discussed above, is the most complicated of these elementary details to present. The other fundamentals are listed in the table below, with some brief remarks and definitions to complete the initial description of this algebraic system.

Quatro-Quaternion Rules: $\forall\, a,b,c \in \mathbb{QQ} \equiv M_{[\cdot,\times]}(2,\mathbb{H});\ \exists\, 0,1,\ a^{-1},\ a^{-1}_{l\times}, a^{-1}_{r\times} \in \mathbb{QQ}$ *s.t.*

rule	+ op	· op	× op
closure	$a+b \in \mathbb{QQ}$	$a \cdot b \in \mathbb{QQ}$	$a \times b \in \mathbb{QQ}$
associativity	$a+(b+c)=(a+b)+c$	$a\cdot(b\cdot c)=(a\cdot b)\cdot c$	$a\times(b\times c)\neq(a\times b)\times c$
identity	$a+0=0+a=a$	$a\cdot 1=1\cdot a=a$	$a\times 1=1\times a=a$
zero	0	$a\cdot 0=0\cdot a=0$	$a\times 0=0\times a=0$
inverse	$a+(-a)=(-a)+a=0$	$a\cdot a^{-1}=a^{-1}\cdot a=1$	$a\times a^{-1}_{r\times}=a^{-1}_{l\times}\times a=1$
commutativity	$a+b=b+a$	$a\cdot b\neq b\cdot a$	$a\times b\neq b\times a$
left	distributivity	$a\cdot(b+c)=(a\cdot b)+(a\cdot c)$	$a\times(b+c)=(a\times b)+(a\times c)$
right	distributivity	$(a+b)\cdot c=(a\cdot c)+(b\cdot c)$	$(a+b)\times c=(a\times c)+(b\times c)$
mixed	associativity	$a\cdot(b\times c)\neq(a\cdot b)\times c$	$a\times(b\cdot c)\neq(a\times b)\cdot c$
left	alternativity	$a\cdot(a\cdot b)=(a\cdot a)\cdot b$	$a\times(a\times b)\neq(a\times a)\times b$
right	alternativity	$(a\cdot b)\cdot b=a\cdot(b\cdot b)$	$(a\times b)\times b\neq a\times(b\times b)$
third	alternativity	$a\cdot(b\cdot a)=(a\cdot b)\cdot a$	$a\times(b\times a)\neq(a\times b)\times a$
conjugation	$(a+b)^*=a^*+b^*$	$(a\cdot b)^*=b^*\cdot a^*$	$(a\times b)^*=b^*\times a^*$
square norm	$N(a)$	$=\ \mathrm{TR}(a\cdot a^*)\ =\ \mathrm{TR}(a^*\cdot a)$	$=\ \mathrm{TR}(a\times a^*)\ =\ \mathrm{TR}(a^*\times a)$
inverse formula	$=-a$	$a^{-1}=(48);$	$a^{-1}_{l\times}=(91);\ \ a^{-1}_{r\times}=(90)$

`conjugation:` any quatro-quaternion, $h \in \mathbb{QQ}$, can be written in the quadruple form, $h=(A,B,C,D)$, which is taken to be equivalent to the following matrix form with the particular parameter order; and a conjugation operator * is then defined by, $h^*=(A^*,C^*,B^*,D^*)$, equivalent to conjugating the quaternions and transposing the matrix form,

$$h=\begin{pmatrix}A & C\\ B & D\end{pmatrix}, \qquad h^*=\begin{pmatrix}A^* & B^*\\ C^* & D^*\end{pmatrix} \qquad A,B,C,D\in\mathbb{H};\ h,h^*\in\mathbb{QQ} \tag{96}$$

The quatro-quaternion system is a “ `dual-product` ” matrix algebra. It has a few notable structural properties. First of all, the two products, $\cdot$ and $\times$, share the same identity element, **1**, which is the 2×2 unit matrix. But, they have different multiplicative inverses. One product $\cdot$ is associative, while the other $\times$ is non-associative. Yet, both these products are separately distributive over $+$ addition. Neither product commutes, and $h\cdot h^*\neq h^*\cdot h$, $h\times h^*\neq h^*\times h$, but these matrix products all have the same trace, so we define that to be a metric square norm. While octonions can be represented as a subalgebra restricted to the *twisted product* $\times$, the $\mathbb{QQ}$ algebra is not itself alternative in this operator. For $o\in\mathbb{QQ}$, with octonion form, $N(o)=2|o|^2$, where $|o|^2$ is the octonion norm.

THE DIFFERENCE OF SQUARES IS A MATRIX OF COMMUTATORS:
When o is a quatro-quaternion, with the same form of an octonion, then there's also an interesting expression for the difference of the non-associative and associative square products, for then $o\times o-o\cdot o$ is a diagonal matrix of the commutators $[A,B]=(AB-BA)$ and $[A,B^*]=(AB^*-B^*A)$.

$$o=\begin{pmatrix}A & -B^*\\ B & A^*\end{pmatrix}, \qquad o\times o-o\cdot o\ =\begin{pmatrix}0 & AB^*-B^*A\\ AB-BA & 0\end{pmatrix} \qquad A,B\in\mathbb{H},\ o\in\mathbb{QQ} \tag{97}$$

Now $[A,B]=0 \iff [A,B^*]=0$, so if the quaternion parameters that define the octonion form commute, then this difference of squares vanishes, and $o\times o=o\cdot o$, and these octonion form numbers then have identical squares for associative and non-associative products. This is also obvious from the formal definitions, (1) and (2), which differ only by simple *twisting*, so commuting parameters produce these equal squares. Conversely, however, if the parameters non-commute, the two product operators *must* produce different squares. In the more general case, for the quatro-quaternion quadruple, $h=(A,B,C,D)$, the difference of squares is the matrix with *three* commutators, $[A,B],[B,C],[C,A]$, the last quaternion, D, not appearing in the result.

$$h=\begin{pmatrix}A & C\\ B & D\end{pmatrix}, \qquad h\times h-h\cdot h=\begin{pmatrix}BC-CB & CA-AC\\ AB-BA & 0\end{pmatrix} \qquad A,B,C,D\in\mathbb{H},\ h\in\mathbb{QQ} \tag{98}$$

That is, $h\times h-h\cdot h=(\ [B,C],\ [A,B],\ [C,A],\ 0\ )$.

Multiplicative Inverses. Despite the fact that we write, $a \cdot a^{-1} = a^{-1} \cdot a = 1$, and $a \times a^{-1}_{r\times} = a^{-1}_{l\times} \times a = 1$, in the above table, not every quatro-quaternion has an inverse. But, when a number has a right inverse it also has a left inverse, and visa versa, for the same product, $\cdot$ or $\times$. The right and left inverses are the same for the $\cdot$ product, and generally different for the $\times$ product—but, in numbers with octonion form they are always identical.

norms. Since the $\mathbb{QQ}$ are 16-dimensional numbers, we define a metric norm to be the trace of the non-associative (or associative) product of a number with its conjugate. This is the sum of 16 real number squares, 4 provided by each of the four component quaternions in the quadruple, $h = (A, B, C, D)$.

$$N(h) = \text{TR}(h \cdot h^*) = \text{TR}(h \times h^*) = |A|^2 + |B|^2 + |C|^2 + |D|^2 \tag{99}$$

The octonions being represented by such quatro-quaternions are only 8-dimensional numbers, which we may write in couple form, $o = (A, B)$, or quadruple form, $o = (A, B, -B^*, A^*)$. But, in either case, the number has two norms, its quatro-quaternion metric norm, $N(o) = 2|A|^2 + 2|B|^2$, and the usual octonion norm, $|o|^2 = |A|^2 + |B|^2$, which differ by the factor of 2 because of the difference in the number of degrees of freedom in the two algebras. Despite the fact that $\mathbb{QQ}$ numbers have norms and conjugates, we cannot write, $h^{-1}_{\times} = h^*/N(h)$, since only when the number has octonion form can we divide the conjugate by a norm to obtain the inverse, in this case, $o^{-1}_{\times} = 2o^*/N(o)$. But, the metric norm also does not have that nice law of the squares property, since, in general, $N(h)N(g) \neq N(h \times g)$.

a third product ?:

The missing fourth quaternion, D, in the expression $h \times h - h \cdot h = (\ [B,C],\ [A,B],\ [C,A],\ 0\ )$, suggests to us that the definition of our non-associative product $\times$ may, in some sense, be incomplete. There is obviously a strange asymmetry here, which intuitively gives one the feeling that something is lacking in our algebra. Now, we can represent octonions just fine with our matrix algebra, as it is, but it's the quatro-quaternion algebra itself that appears to require some kind of fixing. Intuition tells us that if this twisted product $\times$ produces this kind of asymmetry in expressions, maybe there's another twisted product $\overset{\leftharpoondown}{\times}$ that will give the complementary results.

$$\times \;\approx\; \begin{pmatrix} \text{R+L} & \text{L+R} \\ \text{L+R} & \text{R+L} \end{pmatrix} \qquad\qquad \overset{\leftharpoondown}{\times} \;\approx\; \begin{pmatrix} \text{L+R} & \text{R+L} \\ \text{R+L} & \text{L+R} \end{pmatrix}$$

On reviewing our definition for the twisted product, we observe that there's one obvious alteration that complements the definition we selected. There's another way to modify the product expressions to obtain that balance of right and left actions in the sums, and this alternative mirrors the twisting profile that exists in our definition. This suggests that we add another twisted product $\overset{\leftharpoondown}{\times}$ to our algebra, with the complementary definition,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix} \overset{\leftharpoondown}{\times} \begin{pmatrix} B_{00} & B_{01} \\ B_{10} & B_{11} \end{pmatrix} = \begin{pmatrix} B_{00}A_{00} + A_{01}B_{10} & A_{00}B_{01} + B_{11}A_{01} \\ A_{10}B_{00} + B_{10}A_{11} & B_{01}A_{10} + A_{11}B_{11} \end{pmatrix} \tag{100}$$

When we explore the results obtained from using this new product, we find it does indeed provide the missing link. The 4th quaternion, D, re-appears, while it is the 1st quaternion, A, that is now absent from the difference of squares.

$$h = \begin{pmatrix} A & C \\ B & D \end{pmatrix}, \qquad h \overset{\leftharpoondown}{\times} h - h \cdot h = \begin{pmatrix} 0 & DC - CD \\ BD - DB & CB - BC \end{pmatrix} \qquad A, B, C, D \in \mathbb{H},\ h \in \mathbb{QQ} \tag{101}$$

That is, $h \overset{\leftharpoondown}{\times} h - h \cdot h = (\ 0,\ [B,D],\ [D,C],\ [C,B]\ )$.

This leads to the idea of a 3rd operator. However, a close inspection reveals that this operator is so very similar to our existing twisted product in many respects that it is better considered simply a variation of it. The non-associative product itself appears to have a duality, with two forms, one we might term "forward" and the other "reverse." So we use the reversing operator, $\leftharpoondown$, instead, to indicate that every pair of factors in the matrix product definition for that operator be reversed. We can then write, $\overset{\leftharpoondown}{\cdot}$ and $\overset{\leftharpoondown}{\times}$, but shall leave exploration of these for a future work. Suffice it to say, that if $a, b \in \mathbb{QQ}$, have the Cayley-Dickson-(I) octonion form (5), then $a \times b$ also has this octonion form and is constructed following the first product rule for couples defined in (4), but, although $a \overset{\leftharpoondown}{\times} b$ also has octonion form (5), it is not constructed following the first rule, instead it follows the Cayley-Dickson (II)'s second product rule for couples defined in (6), and with the factors in *reverse* order. In other words, if $a = (A, B)$, $b = (C, D)$, are the octonion form quatro-quaternions, then, $a \times b = (A, B) \times (C, D) \equiv (A, B)(C, D)_I$, but, $a \overset{\leftharpoondown}{\times} b = (A, B) \overset{\leftharpoondown}{\times} (C, D) \equiv (C, D)(A, B)_{II}$. So, the alternate product $\overset{\leftharpoondown}{\times}$ reverses the order of the factors AND swaps the Cayley-Dickson processes from (I) to (II).

SCALING FACTORS. Sometimes we'd like to multiply the quatro-quaternion by an overall scaling factor, so we need appropriate definitions to establish this type of operation. Consider, for example, the expressions for the right (90), and left (91), non-associative inverses. If the two divisors within the diagonal matrix are the same, i.e. $D_1 = D_2$, we may write this as one value, D, and thus replace the diagonal matrix with a scaling parameter, like the ordinary matrix algebra case given in (34). These formulas then become,

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}^{-1}_{r\times} = \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} \times \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{D} \end{pmatrix} = \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} \times \frac{1}{D} \tag{102}$$

$$\begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}^{-1}_{l\times} = \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & \frac{1}{D} \end{pmatrix} \times \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} = \frac{1}{D} \times \begin{pmatrix} A_{11} & -A_{01} \\ -A_{10} & A_{00} \end{pmatrix} \tag{103}$$

Note that, in writing the formulas this way, we are effectively extending the definition of multiplication to include the product of a matrix with a "quaternion scaling factor," in (102) and (103). To be consistent with our formal definitions, (1) and (2), we must also allow two different methods for multiplication by such a scaling factor, one without twisting and the other twisted in the very manner that an equivalent diagonal matrix of identical values would produce if the factor were replaced with its diagonal matrix equivalent. This is necessary for a consistent use of the unit matrix.

Consider, $A, B \in \mathbb{H}_R,\ I \in \mathbb{QQ}$, where I is the unit matrix;

$$A \cdot I = A \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot A = I \cdot A \tag{104}$$

$$A \times I = A \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times A = I \times A \tag{105}$$

$$(AB) \times I = (AB) \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & AB \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \times (AB) = I \times (AB) \tag{106}$$

$$A \times (B \times I) = A \times (B \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = A \times \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \times \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & BA \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \times B = (I \times A) \times B \tag{107}$$

etc..then, note that the standard product $\cdot$ obeys the *mixed associative law* with ordinary quaternion multiplication, i.e. $(AB) \cdot I = A \cdot (B \cdot I)$, but the twisted product $\times$ does not associate, $(AB) \times I \neq A \times (B \times I)$. So, one must pay careful attention to the parentheses in the expression when using quaternion scaling factors with the $\times$ product.

Our scaling parameter must always be interchangeable with its corresponding diagonal matrix. Accordingly, we define the two scaling products, $\cdot$ and $\times$, for $\lambda, \alpha, \beta \in \mathbb{H}_R,\ h = (A, B, C, D) \in \mathbb{QQ}$, as follows;

$$\lambda \cdot h = \lambda \cdot \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \cdot \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda A & \lambda C \\ \lambda B & \lambda D \end{pmatrix} \tag{108}$$

$$h \cdot \lambda = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \cdot \lambda = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \cdot \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} A\,\lambda & C\lambda \\ B\lambda & D\lambda \end{pmatrix} \tag{109}$$

$$\lambda \times h = \lambda \times \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \times \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} \lambda A & C\lambda \\ \lambda B & D\lambda \end{pmatrix} \tag{110}$$

$$h \times \lambda = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \times \lambda = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \times \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} A\lambda & C\lambda \\ \lambda B & \lambda D \end{pmatrix} \tag{111}$$

In general, we have, $(\alpha\beta) \cdot h = \alpha \cdot (\beta \cdot h)$, and, $h \cdot (\alpha\beta) = (h \cdot \alpha) \cdot \beta$, but, $(\alpha\beta) \times h \neq \alpha \times (\beta \times h)$, and, $h \times (\alpha\beta) \neq (h \times \alpha) \times \beta$. Also, $(h \times \lambda)^T = \lambda \overleftharpoon{\times} h^T$, and, $(\lambda \times h)^T = h^T \overleftharpoon{\times} \lambda$. We could also write, $\lambda \cdot h = h \overleftharpoon{\cdot} \lambda$, and, $h \cdot \lambda = \lambda \overleftharpoon{\cdot} h$. However, for the general transpose, $(g \times h)^T = h^T \overleftharpoon{\times} g^T$, and, $(g \cdot h)^T = h^T \overleftharpoon{\cdot} g^T,\ \forall\ g, h \in \mathbb{QQ}$. Although we'd like to extend the $\times$ product to quaternions, e.g. to give meaning to $(\alpha \times \beta) \times h$ etc.., the quaternions don't have enough degrees of freedom. However, we shall see later how we can extend the two-hand quaternion algebra with an $\times$ product.

III. TWISTING AND PERCOLATION.

$\mathbb{O} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$? : Our new matrix product allows us to represent octonions by 2×2 matrices over the quaternions. But, the quaternions can also be represented by 2×2 matrices over the complex numbers, and complex numbers represented by 2×2 matrices over reals. So, now we'd like to explore the 4×4 and 8×8 matrix representations of octonions that result from combining these ideas. To do this, we first replace the 4 entries in the 2×2 quatro-quaternion matrix numbers with *general* 2×2 matrices over complex numbers. These 2×2 complex number matrices are therefore not necessarily quaternions, because they span the entire set of $M(2, \mathbb{C})$ numbers. This substitution thus results in a simultaneous generalization of our previous quatro-quaternion algebra, in the process of obtaining our matrix expansion to $M_{[\times]}(4, \mathbb{C})$. Then, when we replace the complex numbers by *general* matrices over reals, spanning the entire set of numbers in $M(2, \mathbb{R})$, we obtain yet a further generalization on expanding to 8×8. And like in the case of the associative algebra of 4×4 matrices over reals, which is equivalent to the tensor product of Hamilton's quaternion algebra with itself, $\mathbb{H} \otimes \mathbb{H} \cong M(4, \mathbb{R})$, we conjecture that, $\mathbb{O} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$, that is, this final extended non-associative algebra is also a "product algebra" of the octonion algebra with itself.

So, how does the twisting action propagate through this matrix expansion?

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} + \begin{pmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{pmatrix}\begin{pmatrix} b_{20} & b_{21} \\ b_{30} & b_{31} \end{pmatrix} & \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} b_{02} & b_{03} \\ b_{12} & b_{13} \end{pmatrix} + \begin{pmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{pmatrix}\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} \\ \begin{pmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{pmatrix}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{20} & b_{21} \\ b_{30} & b_{31} \end{pmatrix} & \begin{pmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{pmatrix}\begin{pmatrix} b_{02} & b_{03} \\ b_{12} & b_{13} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} \end{pmatrix} \tag{112}$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix}\begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix} + \begin{pmatrix} b_{20} & b_{21} \\ b_{30} & b_{31} \end{pmatrix}\begin{pmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{pmatrix} & \begin{pmatrix} b_{02} & b_{03} \\ b_{12} & b_{13} \end{pmatrix}\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} + \begin{pmatrix} a_{02} & a_{03} \\ a_{12} & a_{13} \end{pmatrix}\begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix} \\ \begin{pmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}\begin{pmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{pmatrix} + \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} b_{20} & b_{21} \\ b_{30} & b_{31} \end{pmatrix} & \begin{pmatrix} a_{20} & a_{21} \\ a_{30} & a_{31} \end{pmatrix}\begin{pmatrix} b_{02} & b_{03} \\ b_{12} & b_{13} \end{pmatrix} + \begin{pmatrix} b_{22} & b_{23} \\ b_{32} & b_{33} \end{pmatrix}\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \end{pmatrix} \tag{113}$$

Note that there are two ways to expand our quatro-quaternions. We can use either $M(2, \mathbb{C})$ or $M_{[\times]}(2, \mathbb{C})$ in replacing the four quaternions. But, because the complex numbers commute, it doesn't matter which version we choose, we obtain the same effective 4×4 representation of the octonions. In equations (112) and (113), therefore, we use $M(2, \mathbb{C})$, so that the " internal " 2×2 matrix products use the standard matrix product formula defined in (1), for both $\cdot$ and $\times$ expansions.

Then, working out the internal matrix products and removing the inside parentheses we obtain,

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \cdot \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \qquad \approx \qquad \begin{pmatrix} \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} \\ \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} \\ \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} \\ \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} & \mathrm{R+R+R+R} \end{pmatrix} \tag{114}$$

$$\begin{pmatrix} a_{00}b_{00} + a_{01}b_{10} + a_{02}b_{20} + \boldsymbol{a_{03}b_{30}} & a_{00}b_{01} + a_{01}b_{11} + \boldsymbol{a_{02}b_{21}} + \boldsymbol{a_{03}b_{31}} & a_{00}b_{02} + \boldsymbol{a_{01}b_{12}} + a_{02}b_{22} + a_{03}b_{32} & \boldsymbol{a_{00}b_{03}} + \boldsymbol{a_{01}b_{13}} + a_{02}b_{23} + a_{03}b_{33} \\ a_{10}b_{00} + a_{11}b_{10} + \boldsymbol{a_{12}b_{20}} + \boldsymbol{a_{13}b_{30}} & a_{10}b_{01} + a_{11}b_{11} + \boldsymbol{a_{12}b_{21}} + a_{13}b_{31} & \boldsymbol{a_{10}b_{02}} + \boldsymbol{a_{11}b_{12}} + a_{12}b_{22} + a_{13}b_{32} & \boldsymbol{a_{10}b_{03}} + a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{20}b_{00} + \boldsymbol{a_{21}b_{10}} + a_{22}b_{20} + a_{23}b_{30} & \boldsymbol{a_{20}b_{01}} + \boldsymbol{a_{21}b_{11}} + a_{22}b_{21} + a_{23}b_{31} & a_{20}b_{02} + a_{21}b_{12} + a_{22}b_{22} + \boldsymbol{a_{23}b_{32}} & a_{20}b_{03} + a_{21}b_{13} + \boldsymbol{a_{22}b_{23}} + \boldsymbol{a_{23}b_{33}} \\ \boldsymbol{a_{30}b_{00}} + \boldsymbol{a_{31}b_{10}} + a_{32}b_{20} + a_{33}b_{30} & \boldsymbol{a_{30}b_{01}} + a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{30}b_{02} + a_{31}b_{12} + \boldsymbol{a_{32}b_{22}} + \boldsymbol{a_{33}b_{32}} & a_{30}b_{03} + a_{31}b_{13} + \boldsymbol{a_{32}b_{23}} + a_{33}b_{33} \end{pmatrix}$$

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \qquad \approx \qquad \begin{pmatrix} \mathrm{R+R+L+}\mathbf{L} & \mathrm{R+R+}\mathbf{L}+\mathbf{L} & \mathrm{L+}\mathbf{L}\mathrm{+R+R} & \mathbf{L}+\mathbf{L}\mathrm{+R+R} \\ \mathrm{R+R+}\mathbf{L}+\mathbf{L} & \mathrm{R+R+}\mathbf{L}\mathrm{+L} & \mathbf{L}+\mathbf{L}\mathrm{+R+R} & \mathbf{L}\mathrm{+L+R+R} \\ \mathrm{L+}\mathbf{L}\mathrm{+R+R} & \mathbf{L}+\mathbf{L}\mathrm{+R+R} & \mathrm{R+R+L+}\mathbf{L} & \mathrm{R+R+}\mathbf{L}+\mathbf{L} \\ \mathbf{L}+\mathbf{L}\mathrm{+R+R} & \mathbf{L}\mathrm{+L+R+R} & \mathrm{R+R+}\mathbf{L}+\mathbf{L} & \mathrm{R+R+}\mathbf{L}\mathrm{+L} \end{pmatrix} \tag{115}$$

$$\begin{pmatrix} a_{00}b_{00} + a_{01}b_{10} + b_{20}a_{02} + \boldsymbol{b_{21}a_{12}} & a_{00}b_{01} + a_{01}b_{11} + \boldsymbol{b_{20}a_{03}} + \boldsymbol{b_{21}a_{13}} & b_{02}a_{00} + \boldsymbol{b_{03}a_{10}} + a_{02}b_{22} + a_{03}b_{32} & \boldsymbol{b_{02}a_{01}} + \boldsymbol{b_{03}a_{11}} + a_{02}b_{23} + a_{03}b_{33} \\ a_{10}b_{00} + a_{11}b_{10} + \boldsymbol{b_{30}a_{02}} + \boldsymbol{b_{31}a_{12}} & a_{10}b_{01} + a_{11}b_{11} + \boldsymbol{b_{30}a_{03}} + b_{31}a_{13} & \boldsymbol{b_{12}a_{00}} + \boldsymbol{b_{13}a_{10}} + a_{12}b_{22} + a_{13}b_{32} & \boldsymbol{b_{12}a_{01}} + b_{13}a_{11} + a_{12}b_{23} + a_{13}b_{33} \\ b_{00}a_{20} + \boldsymbol{b_{01}a_{30}} + a_{22}b_{20} + a_{23}b_{30} & \boldsymbol{b_{00}a_{21}} + \boldsymbol{b_{01}a_{31}} + a_{22}b_{21} + a_{23}b_{31} & a_{20}b_{02} + a_{21}b_{12} + b_{22}a_{22} + \boldsymbol{b_{23}a_{32}} & a_{20}b_{03} + a_{21}b_{13} + \boldsymbol{b_{22}a_{23}} + \boldsymbol{b_{23}a_{33}} \\ \boldsymbol{b_{10}a_{20}} + \boldsymbol{b_{11}a_{30}} + a_{32}b_{20} + a_{33}b_{30} & \boldsymbol{b_{10}a_{21}} + b_{11}a_{31} + a_{32}b_{21} + a_{33}b_{31} & a_{30}b_{02} + a_{31}b_{12} + \boldsymbol{b_{32}a_{22}} + \boldsymbol{b_{33}a_{32}} & a_{30}b_{03} + a_{31}b_{13} + \boldsymbol{b_{32}a_{23}} + b_{33}a_{33} \end{pmatrix}$$

The expansion of $\cdot$ in (114) results in the standard formula for the product of two 4×4 matrices. While, the expansion of $\times$ in (115) has a little more going on than just twisting action. In this *derived twisted product*, the entries in the modified matrix product replace the right action sums, R+R+R+R, with twisted products with the profiles, R+R+L+L and L+L+R+R, and the pattern alternates every two columns or rows. But, the most important modification is that some terms move around and change slots, while others are entirely new, and yet others dissappear. The $\boldsymbol{a_{03}b_{30}}$ term moves from the row col $[0, 0]$ position, to the $[1, 1]$ slot location, where it appears in the twisted form $\boldsymbol{b_{30}a_{03}}$. We refer to this phenomena as *percolation*. The percolating terms are marked in **boldface**.

All the terms that percolate from another part of the matrix are also twisted in their new positions. There are 64 terms in the product expression that can move around and change slots. Each slot takes 4 of these terms to compute a sum. So, there are many ways to percolate a matrix product using a permutation modification. However, the other type of percolation that occurs is the replacement of existing terms with new terms that don't even appear in the standard matrix product. Each term is constructed from a pair of a and b factors, and there are $16 \times 16 = 256$ such possible terms to choose from (if we ignore the order of the factors), only 64 of which are used at any one time in the product expression. The remaining $256 - 64 = 192$ terms are available to be swapped into the matrix to replace one or more of the existing terms there. This type of phenomena is exhibited by the two consecutive $\boldsymbol{a_{02}b_{21}} + \boldsymbol{a_{03}b_{31}}$ terms that vanish from slot $[0,1]$ of the standard matrix product, and are replaced by $\boldsymbol{b_{20}a_{03}} + \boldsymbol{b_{21}a_{13}}$ in the *derived twisted product* expression. Since new terms are created and brought into the expression, annihilating the existing terms that they replace, we refer to this type of modification as *generative percolation*, in contrast to the *permuting percolation* that just causes existing terms to jump slots.

Clearly, armed with *twisting* and *percolation*, we have a good many ways to modify the definition of a matrix product. But, the particular modifications that interest us here are those suggested by the need to satisfy the Cayley-Dickson construction process. So, our definition of the $\times$ matrix operator, in this 4×4 matrix algebra over complex numbers, i.e. $M_{[\times]}(4, \mathbb{C})$, will be the expression given in (115). An octonion is then the 4×4 matrix with the form,

$$o = \begin{pmatrix} a & -b^* & -c^* & -d^* \\ b & a^* & d & -c \\ c & -d^* & a^* & b^* \\ d & c^* & -b & a \end{pmatrix}, \qquad a, b, c, d \in \mathbb{C} \tag{116}$$

The octonion conjugate o^* is obtained by transposing this matrix and conjugating the complex numbers. We can then show that, $o \times o^* = (aa^* + bb^* + cc^* + dd^*) \cdot \mathbf{1}$, where $\mathbf{1}$ is the unit 4×4 matrix.

Using the $M(2, \mathbb{R})$ to replace the complex numbers in (115), we obtain the corresponding *derived twisted product* definition for the $\times$ operator for our 8×8 matrix algebra, $M_{[\times]}(8, \mathbb{R})$. An octonion is then the 8×8 matrix,

$$o = \begin{pmatrix} a & -b & -c & -d & -e & -f & -g & -h \\ b & a & d & -c & f & -e & h & -g \\ c & -d & a & b & g & -h & -e & f \\ d & c & -b & a & h & g & -f & -e \\ e & -f & -g & -h & a & b & c & d \\ f & e & h & -g & -b & a & -d & c \\ g & -h & e & f & -c & d & a & -b \\ h & g & -f & e & -d & -c & b & a \end{pmatrix} \qquad a, b, c, d, e, f, g, h \in \mathbb{R} \tag{117}$$

The octonion conjugate o^* is obtained by simply transposing this matrix. For reference, the definition of this product is given in APPENDIX A . It's easily shown that, $o \times o^* = (a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2) \cdot \mathbf{1}$, where $\mathbf{1}$ is the 8×8 unit matrix. These octonion matrix forms follow the Cayley-Dickson (I) construction defined in (4).

The above methods allow us to represent octonions by 4×4 complex, and 8×8 real matrices. However, we cannot use the definitions for the *derived twisted products* given above for the general Cayley-Dickson algebras, since these work only for octonions. If we wish to represent other algebras in the sequence higher than octonions, by 4×4 and 8×8 matrices, over the corresponding previous algebras in the sequence, we must be careful to apply the original twisted product definition for $\times$ given in (2) for all the internal matrices when constructing our matrix expansions.

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \qquad \approx \qquad \begin{pmatrix} \mathrm{R+L+L+}\mathbf{R} & \mathrm{L+R+}\mathbf{R}\mathrm{+}\mathbf{L} & \mathrm{L+}\mathbf{R}\mathrm{+R+L} & \mathbf{R}\mathrm{+}\mathbf{L}\mathrm{+L+R} \\ \mathrm{L+R+}\mathbf{R}\mathrm{+}\mathbf{L} & \mathrm{R+L+}\mathbf{L}\mathrm{+R} & \mathbf{R}\mathrm{+}\mathbf{L}\mathrm{+L+R} & \mathbf{L}\mathrm{+R+R+L} \\ \mathrm{L+}\mathbf{R}\mathrm{+R+L} & \mathbf{R}\mathrm{+}\mathbf{L}\mathrm{+L+R} & \mathrm{R+L+L+}\mathbf{R} & \mathrm{L+R+}\mathbf{R}\mathrm{+}\mathbf{L} \\ \mathbf{R}\mathrm{+}\mathbf{L}\mathrm{+L+R} & \mathbf{L}\mathrm{+R+R+L} & \mathrm{L+R+}\mathbf{R}\mathrm{+}\mathbf{L} & \mathrm{R+L+}\mathbf{L}\mathrm{+R} \end{pmatrix} \tag{118}$$

$$\begin{pmatrix} a_{00}b_{00} + b_{10}a_{01} + b_{20}a_{02} + \boldsymbol{a_{12}b_{21}} & b_{01}a_{00} + a_{01}b_{11} + \boldsymbol{a_{03}b_{20}} + \boldsymbol{b_{21}a_{13}} & b_{02}a_{00} + \boldsymbol{a_{10}b_{03}} + a_{02}b_{22} + b_{32}a_{03} & \boldsymbol{a_{01}b_{02}} + \boldsymbol{b_{03}a_{11}} + b_{23}a_{02} + a_{03}b_{33} \\ b_{00}a_{10} + a_{11}b_{10} + \boldsymbol{a_{02}b_{30}} + \boldsymbol{b_{31}a_{12}} & a_{10}b_{01} + b_{11}a_{11} + \boldsymbol{b_{30}a_{03}} + a_{13}b_{31} & \boldsymbol{a_{00}b_{12}} + \boldsymbol{b_{13}a_{10}} + b_{22}a_{12} + a_{13}b_{32} & \boldsymbol{b_{12}a_{01}} + a_{11}b_{13} + a_{12}b_{23} + b_{33}a_{13} \\ b_{00}a_{20} + \boldsymbol{a_{30}b_{01}} + a_{22}b_{20} + b_{30}a_{23} & \boldsymbol{a_{21}b_{00}} + \boldsymbol{b_{01}a_{31}} + b_{21}a_{22} + a_{23}b_{31} & a_{20}b_{02} + b_{12}a_{21} + b_{22}a_{22} + \boldsymbol{a_{32}b_{23}} & b_{03}a_{20} + a_{21}b_{13} + \boldsymbol{a_{23}b_{22}} + \boldsymbol{b_{23}a_{33}} \\ \boldsymbol{a_{20}b_{10}} + \boldsymbol{b_{11}a_{30}} + b_{20}a_{32} + a_{33}b_{30} & \boldsymbol{b_{10}a_{21}} + a_{31}b_{11} + a_{32}b_{21} + b_{31}a_{33} & b_{02}a_{30} + a_{31}b_{12} + \boldsymbol{a_{22}b_{32}} + \boldsymbol{b_{33}a_{32}} & a_{30}b_{03} + b_{13}a_{31} + \boldsymbol{b_{32}a_{23}} + a_{33}b_{33} \end{pmatrix}$$

The 4×4 matrix expansion shown in (118) results when using $M_{[\times]}(2, \mathbb{C})$ instead of the $M(2, \mathbb{C})$ employed in (115). This now has the correct form for all Cayley-Dickson algebras, not just octonions. Note the percolation is effectively the same, but a different twisting profile appears in the product expression. The twisting profile of some percolated terms are also different. In the expansion (115) all percolated terms appear in left action (**L**) form "$b.a$", but here some are right action (**R**), "$a.b$", also. If therefore one made a distinction between the "$a.b$" and "$b.a$" forms, say because the a and b factors don't commute, one could consider this a different percolation profile. But, in the octonions case the twisting has no effect, so the percolation profile of (118) and (115) are effectively the same. There are thus two ways to construct *derived twisted products* for the octonions, and the forms of the expressions look different, when we maintain the order of the factors in the terms, but they result in the very same representation. The standard product's right action twisting profile, R+R+R+R, is modified now into the balanced patterns, R+L+L+R and L+R+R+L, instead of the previous R+R+L+L and L+L+R+R found in (115). These differences all become important when representing the higher dimensional algebras. However, *twisting* and *percolation* are the only types of modification requried to describe the changes to the matrix product definitions.

We could "invent" other modifications, of course, like sign changes, making + signs into − signs, for example, R+R+R+R into R−R−R+R, etc., or alter the number of terms that make up a sum in one or more slots of the matrix, like changing, R+R+R+R into R+R+R, etc., or include conjugation operations in the expressions, like $A_{00}B^*_{00} + (B_{10}A_{01})^*$, etc.., or perhaps inverses of parameters instead, like $A^{-1}_{00}B_{00} + B^{-1}_{10}A_{01}$, or even extend the algebra with two-hand quaternions, like $A'_{00}B_{00} + B'_{10}A_{01}$, including the hand changing operator, and so on. But these are rather arbitrary, and are not generally suggested by the Cayley-Dickson construction (although one could consider incorporating some particular combination of *conjugation* together with *twisting* and *sign changes* to mimic the C-D process directly into the matrix product definition itself). The modifications we prefer are those that meet both criteria of *the principle of simplicity* and being *suggested by some natural process.* For example, it is reasonably obvious that it is impossible to represent octonions by 4×4 matrices over the complex numbers using a matrix product constructed by *twisting* alone. That is, one must use some form of *percolation.* Although, it is possible that there are several ways to percolate the matrix product expression to obtain alternate forms that can represent these octonions. Since there are only a finite number of ways to twist and percolate, questions like this can be computationally determined by an exhaustive search through the alternatives, when an easier corresponding theoretical proof is unavailable.

MATRIX REPRESENTATIONS OF OCTONION PRODUCT ALGEBRAS

We now have 2×2, 4×4, and 8×8 matrix representations of the octonions, $\mathbb{O}$. But, in the process we've also constructed generalized matrix algebras to facilitate these octonion representations, and these matrix algebras contain more than just the simple octonion algebra. An inspection of the number of degrees of freedom involved in each case suggests that these generalized matrix algebras may be isomorphic to product algebras formed with the octonion algebra, that is, $\mathbb{C} \times \mathbb{O} \cong M_{[\times]}(2, \mathbb{H})$, $\mathbb{H} \times \mathbb{O} \cong M_{[\times]}(4, \mathbb{C})$, $\mathbb{O} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$. So, let us then examine these ideas.

`Conjecture:` $\mathbb{C} \times \mathbb{O} \cong M_{[\times]}(2, \mathbb{H})$. i.e. the product algebra of the complex algebra with the octonion algebra is isomorphic to the non-associative 2×2 matrix algebra over the quaternions defined by the *twisted product* $\times$ given in formal definition (2).
`Proof:`
Let an octonion be, $o = o_u e_u$, with basis, e_u, $u = 0, 1, 2, ..., 7$, and complex number, $z = x + \boldsymbol{i}y$. First we erect a basis, using the 2×2 matrix representation (5) for the octonion, then pick a linearly independent complex $\boldsymbol{i}$ number.

OCTONION:

$$o = \begin{pmatrix} A & -B^* \\ B & A^* \end{pmatrix} = \begin{pmatrix} a+bi+cj+dk & -e+fi+gj+hk \\ e+fi+gj+hk & a-bi-cj-dk \end{pmatrix} \qquad \texttt{where,}\ \begin{matrix} A = a+bi+cj+dk, \\ B = e+fi+gj+hk. \end{matrix} \in \mathbb{H}.$$

$$= a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + c\begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} + d\begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix} + e\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + f\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + g\begin{pmatrix} 0 & j \\ j & 0 \end{pmatrix} + h\begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix} \tag{119}$$

$$= ae_0 + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7 \qquad a,b,c,d,e,f,g,h \in \mathbb{R}$$

COMPLEX NUMBER:

$$z = \begin{pmatrix} x+iy & 0 \\ 0 & x+iy \end{pmatrix} = x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + y\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \tag{120}$$

$$= x + \boldsymbol{i}y \qquad x, y \in \mathbb{R}$$

It's trivial to show that, $\boldsymbol{i}_\times^2 = \boldsymbol{i} \times \boldsymbol{i} = -\mathbf{1}$, where the unit 2×2 matrix is $\mathbf{1} = e_0$. So, the $x + \boldsymbol{i}y$ of (120) is isomorphic to the complex numbers, and since the only octonion basis element with the quaternion unit i in the main diagonal is e_1, where the entries appear with opposite signs, there's no way to construct the chosen complex $\boldsymbol{i}$ number by a linear combination of the eight octonion basis elements. So, our $\boldsymbol{i}$ is linearly independent of the octonion basis. By usual convention, we sometimes suppress the explicit depiction of the unit matrix, so that we can write, $x\mathbf{1} = x$. Also, despite the fact that we have only defined $\cdot$ and $\times$ operations between quaternion scaling factors and quatro-quaternions (108) − (111), when the scaling factor is a simple real number scalar there's no difference between the two product forms, and so, we now adopt the usual convention of representing this product by simple juxtaposition of parameters: i.e. $\lambda \times e_u = e_u \times \lambda = \lambda \cdot e_u = e_u \cdot \lambda = \lambda e_u = e_u \lambda, \ \forall\, \lambda \in \mathbb{R}$, etc..

But, we need to establish that these eight 2×2 matrices in (119) do form an octonion basis. To confirm this, we take the *twisted product* of pairs of matrices, $e_u \times e_v, \ u, v = 0, 1, 2, ..., 7$, where we obtain the following product table:

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	e_3	$-e_2$	$-e_5$	e_4	$-e_7$	e_6
e_2	e_2	$-e_3$	$-e_0$	e_1	$-e_6$	e_7	e_4	$-e_5$
e_3	e_3	e_2	$-e_1$	$-e_0$	$-e_7$	$-e_6$	e_5	e_4
e_4	e_4	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_5	e_5	$-e_4$	$-e_7$	e_6	e_1	$-e_0$	$-e_3$	e_2
e_6	e_6	e_7	$-e_4$	$-e_5$	e_2	e_3	$-e_0$	$-e_1$
e_7	e_7	$-e_6$	e_5	$-e_4$	e_3	$-e_2$	e_1	$-e_0$

$\longrightarrow$

$\times$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	$\overline{0}$	3	$\overline{2}$	$\overline{5}$	4	$\overline{7}$	6
2	2	$\overline{3}$	$\overline{0}$	1	$\overline{6}$	7	4	$\overline{5}$
3	3	2	$\overline{1}$	$\overline{0}$	$\overline{7}$	$\overline{6}$	5	4
4	4	5	6	7	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
5	5	$\overline{4}$	$\overline{7}$	6	1	$\overline{0}$	$\overline{3}$	2
6	6	7	$\overline{4}$	$\overline{5}$	2	3	$\overline{0}$	$\overline{1}$
7	7	$\overline{6}$	5	$\overline{4}$	3	$\overline{2}$	1	$\overline{0}$

(121)

OCTONION PRODUCT TABLE IN THE $M_{[\times]}(2, \mathbb{H})$ REPRESENTATION. & COMPACT FORM.

The left column e_u multiply the top row e_v and the result $e_u \times e_v$ is then shown in the body of table (121). A more compact form of this same table is shown on the right, where we use only the index values of the e_u basis elements, with minus sign indicated by a bar over the index value. While there are several ways to erect octonion bases, and thus construct variations of this product table (480 of them), there are always 7 quaternion triples in the basis set. The table, as it stands, however, is not very revealing, unless one is already very familiar with octonion tables. So, let us re-construct the basis using a more intuitive method, where we can easily see that we've got octonions. Since octonions are created from reals by doubling three times, we only need 3 imaginary elements to describe the basis, one new imaginary each time we double. Let's call them, $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

$$o = A + \boldsymbol{k}B \qquad A, B \in \mathbb{H} \tag{122}$$

$$A = z + \overleftarrow{\boldsymbol{j}w}, \ B = u + \overleftarrow{\boldsymbol{j}v} \qquad z, w, u, v \in \mathbb{C} \tag{123}$$

$$z = a + \boldsymbol{i}b, \ w = c + \boldsymbol{i}d, \ u = e + \boldsymbol{i}f, \ v = g + \boldsymbol{i}h \qquad a, b, c, d, e, f, g, h \in \mathbb{R} \tag{124}$$

$$\therefore \quad o = (a + \boldsymbol{i}b) + (c + \boldsymbol{i}d)\boldsymbol{j} + \boldsymbol{k}(e + \boldsymbol{i}f) + \boldsymbol{k}((g + \boldsymbol{i}h)\boldsymbol{j}) = a + b\boldsymbol{i} + c\boldsymbol{j} + d(\boldsymbol{ij}) + e\boldsymbol{k} + f(\boldsymbol{ki}) + g(\boldsymbol{kj}) + h(\boldsymbol{k}(\boldsymbol{ij})) \tag{125}$$

Simple substitution reveals a basis set $\{\ 1,\ \boldsymbol{i},\ \boldsymbol{j},\ (\boldsymbol{ij}),\ \boldsymbol{k},\ (\boldsymbol{ki}),\ (\boldsymbol{kj}),\ (\boldsymbol{k}(\boldsymbol{ij}))\ \}$. The basis begins with the complex number basis, $\{\ 1,\ \boldsymbol{i}\ \}$, which has the property, $\boldsymbol{i}^2 = -1$. Then, the introduction of the $\boldsymbol{j}$ breaks commutativity, and we have the quaternion basis, $\{\ 1,\ \boldsymbol{i},\ \boldsymbol{j},\ (\boldsymbol{ij})\ \}$, introducing the first quaternion triple, $\{\boldsymbol{i}, \boldsymbol{j}, (\boldsymbol{ij})\}$, adding property, $\boldsymbol{j}^2 = -1$, and anti-commuting law, $\boldsymbol{ij} = -\boldsymbol{ji}$, from which we derive, $(\boldsymbol{ij})^2 = (\boldsymbol{ij})(\boldsymbol{ij}) = -(\boldsymbol{ij})(\boldsymbol{ji}) = -\boldsymbol{i}(\boldsymbol{jj})\boldsymbol{i} = \boldsymbol{ii} = -1$, using the still intact associativity law for quaternions, which also gets us the remaining anti-commuting relations, $(\boldsymbol{ij})\boldsymbol{i} = -\boldsymbol{i}(\boldsymbol{ij}) = \boldsymbol{j}$, and, $\boldsymbol{j}(\boldsymbol{ij}) = -(\boldsymbol{ij})\boldsymbol{j} = \boldsymbol{i}$.

$\cdot$	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$(\boldsymbol{ij})$	$\boldsymbol{k}$	$(\boldsymbol{ki})$	$(\boldsymbol{kj})$	$(\boldsymbol{k}(\boldsymbol{ij}))$
1	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$(\boldsymbol{ij})$	$\boldsymbol{k}$	$(\boldsymbol{ki})$	$(\boldsymbol{kj})$	$(\boldsymbol{k}(\boldsymbol{ij}))$
$\boldsymbol{i}$	$\boldsymbol{i}$	-1	$(\boldsymbol{ij})$	$-\boldsymbol{j}$	$-(\boldsymbol{ki})$	$\boldsymbol{k}$	$-(\boldsymbol{k}(\boldsymbol{ij}))$	$(\boldsymbol{kj})$
$\boldsymbol{j}$	$\boldsymbol{j}$	$-(\boldsymbol{ij})$	-1	$\boldsymbol{i}$	$-(\boldsymbol{kj})$	$(\boldsymbol{k}(\boldsymbol{ij}))$	$\boldsymbol{k}$	$-(\boldsymbol{ki})$
$(\boldsymbol{ij})$	$(\boldsymbol{ij})$	$\boldsymbol{j}$	$-\boldsymbol{i}$	-1	$-(\boldsymbol{k}(\boldsymbol{ij}))$	$-(\boldsymbol{kj})$	$(\boldsymbol{ki})$	$\boldsymbol{k}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$(\boldsymbol{ki})$	$(\boldsymbol{kj})$	$(\boldsymbol{k}(\boldsymbol{ij}))$	-1	$-\boldsymbol{i}$	$-\boldsymbol{j}$	$-(\boldsymbol{ij})$
$(\boldsymbol{ki})$	$(\boldsymbol{ki})$	$-\boldsymbol{k}$	$-(\boldsymbol{k}(\boldsymbol{ij}))$	$(\boldsymbol{kj})$	$\boldsymbol{i}$	-1	$-(\boldsymbol{ij})$	$\boldsymbol{j}$
$(\boldsymbol{kj})$	$(\boldsymbol{kj})$	$(\boldsymbol{k}(\boldsymbol{ij}))$	$-\boldsymbol{k}$	$-(\boldsymbol{ki})$	$\boldsymbol{j}$	$(\boldsymbol{ij})$	-1	$-\boldsymbol{i}$
$(\boldsymbol{k}(\boldsymbol{ij}))$	$(\boldsymbol{k}(\boldsymbol{ij}))$	$-(\boldsymbol{kj})$	$(\boldsymbol{ki})$	$-\boldsymbol{k}$	$(\boldsymbol{ij})$	$-\boldsymbol{j}$	$\boldsymbol{i}$	-1

(126)

A THREE GENERATORS $\boldsymbol{ijk}$ OCTONION PRODUCT TABLE.

This allows us to establish the upper quarter of the product table (126). Finally, the introduction of the third element, $\boldsymbol{k}$, breaks associativity, and gives us the remaining four imaginaries, $\{\ \boldsymbol{k},\ (\boldsymbol{ki}),\ (\boldsymbol{kj}),\ (\boldsymbol{k}(\boldsymbol{ij}))\ \}$, completing the octonion basis, and adding the defining property, $\boldsymbol{k}^2 = -1$. But, every two octonion imaginary basis elements also form a quaternion triple with their product, so we get second and third triples, $(\boldsymbol{k}, \boldsymbol{i}, (\boldsymbol{ki}))$ and $(\boldsymbol{k}, \boldsymbol{j}, (\boldsymbol{kj}))$, with anti-commuting laws, $(\boldsymbol{ki}) = -(\boldsymbol{ik})$, $(\boldsymbol{kj}) = -(\boldsymbol{jk})$ etc., and since the associative law still holds for such quaternion triples we can reckon in the usual manner, for example, $(\boldsymbol{ki})^2 = (\boldsymbol{ki})(\boldsymbol{ki}) = -(\boldsymbol{ik})(\boldsymbol{ki}) = -\boldsymbol{i}(\boldsymbol{kk})\boldsymbol{i} = \boldsymbol{ii} = -1$; and likewise, all other pairs anti-commute, $\boldsymbol{xy} = -\boldsymbol{yx}$, every imaginary has, $\boldsymbol{x}^2 = -1$, so we can immediately fill in the product table's remaining main diagonal elements, $(\boldsymbol{ki})^2 = (\boldsymbol{kj})^2 = (\boldsymbol{k}(\boldsymbol{ij}))^2 = -1$, and effectively need only work out the results for products above this diagonal. The anti-commutative and associative laws for these triples also tell us that, $(\boldsymbol{ki})\boldsymbol{k} = -\boldsymbol{k}(\boldsymbol{ki}) = \boldsymbol{i}$, $\boldsymbol{i}(\boldsymbol{ki}) = -(\boldsymbol{ki})\boldsymbol{i} = \boldsymbol{k}$, and again that, $(\boldsymbol{kj})\boldsymbol{k} = -\boldsymbol{k}(\boldsymbol{kj}) = \boldsymbol{j}$, $\boldsymbol{j}(\boldsymbol{kj}) = -(\boldsymbol{kj})\boldsymbol{j} = \boldsymbol{k}$.

Since there are 7 imaginary basis units in the octonion, there are $7 \times 6/2! = 21$ ways to pick a pair of elements to generate a triple, but each triple requires 3 pairs of elements (e.g. for its three anti-commuting relations), so there are exactly $21/3 = 7$ unique quaternion triples to be found. The fourth triple is, $\{\boldsymbol{k}, (\boldsymbol{ij}), (\boldsymbol{k}(\boldsymbol{ij}))\}$, which gives us the relations, $(\boldsymbol{k}(\boldsymbol{ij}))^2 = -1$, $\boldsymbol{k}(\boldsymbol{ij}) = -(\boldsymbol{ij})\boldsymbol{k}$, and, $(\boldsymbol{ij})(\boldsymbol{k}(\boldsymbol{ij})) = -(\boldsymbol{k}(\boldsymbol{ij}))(\boldsymbol{ij}) = \boldsymbol{k}$, $(\boldsymbol{k}(\boldsymbol{ij}))\boldsymbol{k} = -\boldsymbol{k}(\boldsymbol{k}(\boldsymbol{ij})) = (\boldsymbol{ij})$. But, to progress any further we must now apply the modifications to the law of associativity.

$$
\begin{array}{llllll}
1: & \{e_1, e_2, e_3\} & \boldsymbol{i}, & \boldsymbol{j}, & (\boldsymbol{ij}) \\
2: & \{e_4, e_1, e_5\} & \boldsymbol{k}, & \boldsymbol{i}, & (\boldsymbol{ki}) \\
3: & \{e_4, e_2, e_6\} & \boldsymbol{k}, & \boldsymbol{j}, & (\boldsymbol{kj}) \\
4: & \{e_4, e_3, e_7\} & \boldsymbol{k}, & (\boldsymbol{ij}), & (\boldsymbol{k}(\boldsymbol{ij})) \\
5: & \{e_6, e_5, e_3\} & (\boldsymbol{kj}), & (\boldsymbol{ki}), & (\boldsymbol{ij}) \\
6: & \{e_1, e_7, e_6\} & \boldsymbol{i}, & (\boldsymbol{k}(\boldsymbol{ij})), & (\boldsymbol{kj}) \\
7: & \{e_2, e_5, e_7\} & \boldsymbol{j}, & (\boldsymbol{ki}), & (\boldsymbol{k}(\boldsymbol{ij}))
\end{array} \tag{127}
$$

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & \boldsymbol{i} & \boldsymbol{j} & (\boldsymbol{ij}) & \boldsymbol{k} & (\boldsymbol{ki}) & (\boldsymbol{kj}) & (\boldsymbol{k}(\boldsymbol{ij})) \\
e_0 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7
\end{array} \tag{128}
$$

The three generators $\boldsymbol{ijk}$ obey the anti-associating law, $\boldsymbol{x}(\boldsymbol{yz}) = -(\boldsymbol{xy})\boldsymbol{z}$, where the elements $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, are assigned to the variables $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, in any order. In fact, any three different imaginary basis elements, that do not form a quaternion triple, form an anti-associating triple like these generators. So, we now have, $\boldsymbol{i}(\boldsymbol{kj}) = -\boldsymbol{i}(\boldsymbol{jk}) = (\boldsymbol{ij})\boldsymbol{k} = -\boldsymbol{k}(\boldsymbol{ij})$, and, $\boldsymbol{j}(\boldsymbol{ki}) = -\boldsymbol{j}(\boldsymbol{ik}) = (\boldsymbol{ji})\boldsymbol{k} = -(\boldsymbol{ij})\boldsymbol{k} = \boldsymbol{k}(\boldsymbol{ij})$. Thus, two kinds of sets with 3 distinct elements co-exist among the 7 imaginaries: "associating triples" and "anti-associating triples." If a triple is formed with two identical elements the alternative laws apply—there is a left (first) alternative law, $\boldsymbol{x}(\boldsymbol{xy}) = (\boldsymbol{xx})\boldsymbol{y}$, a right (second) alternative law, $(\boldsymbol{xy})\boldsymbol{y} = \boldsymbol{x}(\boldsymbol{yy})$, and a third alternative law, $\boldsymbol{x}(\boldsymbol{yx}) = (\boldsymbol{xy})\boldsymbol{x}$—and we then have "alternative triples." But, since every two elements generate a quaternion triple, these alternative laws all follow from the anti-commuting and associative laws of the quaternion subalgebras. Conversely, the alternative laws can be used instead to imply the existence of the quaternion triples, and represent the weak form of associativity found in the octonion algebra.

$$
\begin{aligned}
\{\boldsymbol{i}, \boldsymbol{k}, (\boldsymbol{ij})\} \implies \boldsymbol{i}(\boldsymbol{k}(\boldsymbol{ij})) &= -\boldsymbol{i}((\boldsymbol{ij})\boldsymbol{k}) = (\boldsymbol{i}(\boldsymbol{ij}))\boldsymbol{k} = ((\boldsymbol{ii})\boldsymbol{j})\boldsymbol{k} = -(\boldsymbol{jk}) = (\boldsymbol{kj}) && (129)\\
(\boldsymbol{ij})(\boldsymbol{ki}) &= -(\boldsymbol{ij})(\boldsymbol{ik}) = ((\boldsymbol{ij})\boldsymbol{i})\boldsymbol{k} = -((\boldsymbol{ji})\boldsymbol{i})\boldsymbol{k} = -(\boldsymbol{j}(\boldsymbol{ii}))\boldsymbol{k} = (\boldsymbol{jk}) = -(\boldsymbol{kj}) && (130)\\
\{\boldsymbol{j}, \boldsymbol{k}, (\boldsymbol{ij})\} \implies \boldsymbol{j}(\boldsymbol{k}(\boldsymbol{ij})) &= -(\boldsymbol{ki}) && (131)\\
(\boldsymbol{ij})(\boldsymbol{kj}) &= \ \ (\boldsymbol{ki}) && (132)\\
\{\boldsymbol{k}, \boldsymbol{i}, (\boldsymbol{kj})\} \implies (\boldsymbol{ki})(\boldsymbol{kj}) &= -(\boldsymbol{ik})(\boldsymbol{kj}) = \boldsymbol{i}(\boldsymbol{k}(\boldsymbol{kj})) = \boldsymbol{i}((\boldsymbol{kk})\boldsymbol{j}) = -(\boldsymbol{ij}) && (133)\\
\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \implies (\boldsymbol{ki})(\boldsymbol{k}(\boldsymbol{ij})) &= -(\boldsymbol{ki})((\boldsymbol{ki})\boldsymbol{j}) = -((\boldsymbol{ki})(\boldsymbol{ki}))\boldsymbol{j} = \boldsymbol{j} && (134)\\
(\boldsymbol{kj})(\boldsymbol{k}(\boldsymbol{ij})) &= -(\boldsymbol{kj})(\boldsymbol{k}(\boldsymbol{ji})) = (\boldsymbol{kj})((\boldsymbol{kj})\boldsymbol{i}) = ((\boldsymbol{kj})(\boldsymbol{kj}))\boldsymbol{i} = -\boldsymbol{i} && (135)
\end{aligned}
$$

By identifying the appropriate anti-associating triples (given above on the left in (129) – (135)) we can resolve the remaining products to complete table (126). To find the remaining quaternion triples, we need to make use of the anti-associating law, e.g. from (133) we deduce, $(\boldsymbol{kj})(\boldsymbol{ki}) = -(\boldsymbol{ki})(\boldsymbol{kj}) = (\boldsymbol{ij})$, so, $\{(\boldsymbol{kj}), (\boldsymbol{ki}), (\boldsymbol{ij})\}$, is a fifth quaternion triple. The sixth and seventh quaternion triples are given in table (127). A comparison of the table (126) and (121) shows they have the same form, when the label assignments given in (128) are made. We deliberately designed our intuitive method to obtain the basis elements in corresponding sequence, hence the mystery twist in (123).

The ijk basis for the quaternions substituted in (119) are “right-handed,” while the Cayley-Dickson (I) process would give us a “left-hand” quaternion for our initial quaternion triple. Hence, the reverse operator avoids the alternative of having to rearrange the table afterwards, just to recognise the match between (126) and (121).

Now that we have an octonion basis, and independent complex number basis, we take the product of the two bases to expand the set of elements. With the complex $\boldsymbol{i}$ from (120) multiplying from the “left” side of the octonions, we obtain the following $\boldsymbol{i} \times e_u$ set of matrices,

$$\begin{array}{cccccccc} \boldsymbol{i}\times e_0 & \boldsymbol{i}\times e_1 & \boldsymbol{i}\times e_2 & \boldsymbol{i}\times e_3 & \boldsymbol{i}\times e_4 & \boldsymbol{i}\times e_5 & \boldsymbol{i}\times e_6 & \boldsymbol{i}\times e_7 \\ = & = & = & = & = & = & = & = \\ \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} & \begin{pmatrix} -j & 0 \\ 0 & -j \end{pmatrix} & \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix} & \begin{pmatrix} 0 & j \\ -j & 0 \end{pmatrix} \end{array} \tag{136}$$

If we multiply $\boldsymbol{i}$ from the “right” side of the octonions, instead, we obtain a slightly different set, $\boldsymbol{e}_u \times \boldsymbol{i}$, of 2×2 matrices, since two of the products anti-commute, $\boldsymbol{i} \times e_2 = -e_2 \times \boldsymbol{i}$, and, $\boldsymbol{i} \times e_3 = -e_3 \times \boldsymbol{i}$, while all the others commute, $\boldsymbol{i} \times e_u = e_u \times \boldsymbol{i}, \ \ \forall\, u = 0, 1, 4, 5, 6, 7$. However, the 8 matrices in (136), together with the original 8 e_u of (119) form a set of 16 linearly independent basis elements, completely representing $M_{[\times]}(2, \mathbb{H})$.

Therefore, we define a 16-dimension basis, $\beta_u, \ u = 0, 1, 2, \ldots, 9, \text{A}, \text{B}, \ldots, \text{F}$, by, $\beta_u = e_u, \ \beta_{u+8} = \boldsymbol{i} \times e_u, \ u = 0, 1, 2, ..., 7$, where the letters, A, B, …, F, are now the numbers $10 - 15$ in “hexadecimal notation.”

The “compact form” product table for this $\mathbb{C} \times \mathbb{O}$ basis is on the right ⇒

×	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
1	1	$\overline{0}$	3	$\overline{2}$	$\overline{5}$	4	$\overline{7}$	6	9	$\overline{8}$	B	$\overline{A}$	$\overline{D}$	C	$\overline{F}$	E
2	2	$\overline{3}$	$\overline{0}$	1	$\overline{6}$	7	4	$\overline{5}$	$\overline{A}$	B	8	$\overline{9}$	E	$\overline{F}$	$\overline{C}$	D
3	3	2	$\overline{1}$	$\overline{0}$	$\overline{7}$	$\overline{6}$	5	4	$\overline{B}$	$\overline{A}$	9	8	F	E	$\overline{D}$	$\overline{C}$
4	4	5	6	7	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	C	D	E	F	$\overline{8}$	$\overline{9}$	$\overline{A}$	$\overline{B}$
5	5	$\overline{4}$	$\overline{7}$	6	1	$\overline{0}$	$\overline{3}$	2	D	$\overline{C}$	$\overline{F}$	E	9	$\overline{8}$	$\overline{B}$	A
6	6	7	$\overline{4}$	$\overline{5}$	2	3	$\overline{0}$	$\overline{1}$	E	F	$\overline{C}$	$\overline{D}$	A	B	$\overline{8}$	$\overline{9}$
7	7	$\overline{6}$	5	$\overline{4}$	3	$\overline{2}$	1	$\overline{0}$	F	$\overline{E}$	D	$\overline{C}$	B	$\overline{A}$	9	$\overline{8}$
8	8	9	A	B	C	D	E	F	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$	$\overline{4}$	$\overline{5}$	$\overline{6}$	$\overline{7}$
9	9	$\overline{8}$	B	$\overline{A}$	$\overline{D}$	C	$\overline{F}$	E	$\overline{1}$	0	$\overline{3}$	2	5	$\overline{4}$	7	$\overline{6}$
A	A	$\overline{B}$	$\overline{8}$	9	E	$\overline{F}$	$\overline{C}$	D	2	$\overline{3}$	$\overline{0}$	1	6	$\overline{7}$	$\overline{4}$	5
B	B	A	$\overline{9}$	$\overline{8}$	F	E	$\overline{D}$	$\overline{C}$	3	2	$\overline{1}$	$\overline{0}$	7	6	$\overline{5}$	$\overline{4}$
C	C	D	$\overline{E}$	$\overline{F}$	$\overline{8}$	$\overline{9}$	A	B	$\overline{4}$	$\overline{5}$	6	7	0	1	$\overline{2}$	$\overline{3}$
D	D	$\overline{C}$	F	$\overline{E}$	9	$\overline{8}$	B	$\overline{A}$	$\overline{5}$	4	$\overline{7}$	6	$\overline{1}$	0	$\overline{3}$	2
E	E	F	C	D	$\overline{A}$	$\overline{B}$	$\overline{8}$	$\overline{9}$	$\overline{6}$	$\overline{7}$	$\overline{4}$	$\overline{5}$	2	3	0	1
F	F	$\overline{E}$	$\overline{D}$	C	$\overline{B}$	A	9	$\overline{8}$	$\overline{7}$	6	5	$\overline{4}$	3	$\overline{2}$	$\overline{1}$	0

(137)

16-DIM $\mathbb{C} \times \mathbb{O}$ BASIS FOR $M_{[\times]}(2, \mathbb{H})$

$\mathbb{C} \times \mathbb{O}$ “PRODUCT ALGEBRA” VS. $\mathbb{C} \otimes \mathbb{O}$ “TENSOR PRODUCT”

COMPARISON WITH THE TENSOR PRODUCT:

$\mathbb{C} \otimes \mathbb{O}$: The tensor product of two algebras, $\mathcal{A}$ and $\mathcal{B}$, is the algebra, $\mathcal{A} \otimes \mathcal{B}$, with product defined by,

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (a_1 \odot_{\mathcal{A}} a_2) \otimes (b_1 \odot_{\mathcal{B}} b_2) \qquad a_1, a_2 \in \mathcal{A}; \ \ b_1, b_2 \in \mathcal{B}. \tag{138}$$

where $\odot_{\mathcal{A}}$ is the product operator for the $\mathcal{A}$-algebra, and $\odot_{\mathcal{B}}$ is the product operator for the $\mathcal{B}$-algebra. In the product of two elements of $\mathcal{A} \otimes \mathcal{B}$ the $\mathcal{A}$ elements do not multiply the $\mathcal{B}$ elements, whereas in our 16-dimensional basis matrix representation $\mathbb{C} \times \mathbb{O}$ of $M_{[\times]}(2, \mathbb{H})$ the $\mathbb{C}$omplex basis does actually combine with the $\mathbb{O}$ctonion basis.

$$\begin{array}{rcll} & & \mathbb{C} \otimes \mathbb{O} & \qquad M_{[\times]}(2, \mathbb{H}) \\ (1 \otimes e_u)(1 \otimes e_v) & = & (1 \times 1) \otimes (e_u \times e_v) \iff & (1 \times e_u) \times (1 \times e_v) = 1 \times (e_u \times e_v) \\ (1 \otimes e_u)(\boldsymbol{i} \otimes e_v) & = & (1 \times \boldsymbol{i}) \otimes (e_u \times e_v) \iff & e_u \times (\boldsymbol{i} \times e_v) = \boldsymbol{i} \times (e_u \times e_v) \\ (\boldsymbol{i} \otimes e_u)(1 \otimes e_v) & = & (\boldsymbol{i} \times 1) \otimes (e_u \times e_v) \iff & (\boldsymbol{i} \times e_u) \times e_v = \boldsymbol{i} \times (e_u \times e_v) \\ (\boldsymbol{i} \otimes e_u)(\boldsymbol{i} \otimes e_v) & = & (\boldsymbol{i} \times \boldsymbol{i}) \otimes (e_u \times e_v) \iff & (\boldsymbol{i} \times e_u) \times (\boldsymbol{i} \times e_v) = -(e_u \times e_v) \end{array}$$

(139)
(140)
(141)
(142)

In order for these two products to be the same, the two algebras would have to have the same product table, despite the fact that in the one case, $\mathbb{C} \otimes \mathbb{O}$, parameters multiply only their own kind, remaining essentially an ordered couple, $a \otimes b \equiv (a, b), \ a \in \mathbb{C}, \ b \in \mathbb{O}$, while in the other, $\mathbb{C} \times \mathbb{O} \equiv M_{[\times]}(2, \mathbb{H})$, the parameters mix and merge into one. Since we have proven that both the complex algebra and octonion algebra are represented by the twisted product over the 2×2 matrices, we can now use this non-associative matrix $\times$ product for both multiplication operators of these algebras, i.e. $\odot_{\mathbb{C}} \equiv \times$ and $\odot_{\mathbb{O}} \equiv \times$, when using this matrix representation, and the problem is reduced to that of showing the equivalence of two different ways to multiply the three basis units, $\boldsymbol{i}, \boldsymbol{e}_u, \boldsymbol{e}_v$, within the $M_{[\times]}(2, \mathbb{H})$ algebra itself. Essentially, we'd have to show, $(c_1 \times e_u) \times (c_2 \times e_v) = (c_1 \times c_2) \times (e_u \times e_v)$, for an algebra known to be generally non-commutative and non-associative! This is equivalent to proving the identities in (139)-(142).

Now, looking at (140), we easily see that when, $v = 0$, we obtain the requirement, $e_u \times \boldsymbol{i} = \boldsymbol{i} \times e_u$, and we already know that this fails when, $u = 2, 3$. So, this simple counter example demonstrates that the two are not the same. That is, the "product algebra", $\mathbb{C} \times \mathbb{O}$, differs from the "tensor product", $\mathbb{C} \otimes \mathbb{O}$.

But, the difference is only reflected in the *sign changes* between the product tables. If we define a corresponding tensor product basis, $\mu_v = 1 \otimes e_v$, $\mu_{v+8} = \boldsymbol{i} \otimes e_v$, $v = 0, 1, \ldots, 7$, then construct the product table for, $\mathbb{C} \otimes \mathbb{O}$, we find the $\mu_u \mu_v$ products all have identical values to the $\beta_u \times \beta_v$ products shown in table (137) up to a simple sign change. There are many sign changes, however, since 64 signs need to flip in this $\mathbb{C} \times \mathbb{O}$ table to obtain the corresponding table for $\mathbb{C} \otimes \mathbb{O}$. The values that flip sign are shown on the right ⇒ The dots " . " in the table indicate both the values and signs are identical between the tables (143) and (137). Although octonion tables can have many variations, with the seven quaternion triples introducing various sign flips when exchanging R-H with L-H forms, there's no re-arrangement of basis elements that gets the tables to match, because the two structures, $\mathbb{C} \otimes \mathbb{O}$ and $\mathbb{C} \times \mathbb{O}$, are so very different. So, even though the sign flips indicate an obvious real linear map between the "products," $\lambda\colon \beta_u \times \beta_v \mapsto \mu_u \mu_v = \lambda_{uv} \beta_u \times \beta_v$, $\lambda_{uv} \in \{+1, -1\} \subset \mathbb{R}$, there's no corresponding real linear map between the algebras, $\mathbb{C} \times \mathbb{O} \mapsto \mathbb{C} \otimes \mathbb{O}$, that would facilitate this relationship.

$$M_{[\times]}(2, \mathbb{H}) \;\cong\; \mathbb{O} \,+\, i \times \mathbb{O}. \qquad \text{"\textit{Twisted Bi-Octonions}"}$$

$$\mathbb{C} \otimes \mathbb{O} \;\cong\; \mathbb{O} \,+\, i\mathbb{O}. \qquad \text{"\textit{Bi-Octonions}"}$$

×	0	1	2	3	4	5	6	7	8	9	A	B	C	D	E	F
0	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
1	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
2	.	.	.	.	.	.	.	.	A	$\bar{B}$	$\bar{8}$	9	$\bar{E}$	F	C	$\bar{D}$
3	.	.	.	.	.	.	.	.	B	A	$\bar{9}$	$\bar{8}$	$\bar{F}$	$\bar{E}$	D	C
4	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
5	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
6	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
7	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
8	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
9	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.
A	.	.	.	.	$\bar{E}$	F	C	$\bar{D}$	$\bar{2}$	3	0	$\bar{1}$	.	.	.	.
B	.	.	.	.	$\bar{F}$	$\bar{E}$	D	C	$\bar{3}$	$\bar{2}$	1	0	.	.	.	.
C	.	.	E	F	.	.	$\bar{A}$	$\bar{B}$	.	.	$\bar{6}$	$\bar{7}$	.	.	2	3
D	.	.	$\bar{F}$	E	.	.	$\bar{B}$	A	.	.	7	$\bar{6}$	.	.	3	$\bar{2}$
E	.	.	$\bar{C}$	$\bar{D}$	A	B	.	.	.	.	4	5	$\bar{2}$	$\bar{3}$	.	.
F	.	.	D	$\bar{C}$	B	$\bar{A}$	.	.	.	.	$\bar{5}$	4	$\bar{3}$	2	.	.

(143)

THE $\mathbb{C} \otimes \mathbb{O}$ DIFFERENCES FROM $\mathbb{C} \times \mathbb{O}$

Any $h \in M_{[\times]}(2, \mathbb{H})$ can be written in terms of the $\mathbb{C} \times \mathbb{O}$ basis elements β_u, e.g.,

$$h \;=\; \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \begin{pmatrix} a_0 + a_1 i + a_2 j + a_3 k & c_0 + c_1 i + c_2 j + c_3 k \\ b_0 + b_1 i + b_2 j + b_3 k & d_0 + d_1 i + d_2 j + d_3 k \end{pmatrix} = \sum_{u=0}^{15} \lambda_u \beta_u \qquad A, B, C, D \in \mathbb{H}, \;\; a_s, b_s, c_s, d_s, \lambda_u \in \mathbb{R} \tag{144}$$

$$\begin{aligned} &= \frac{(a_0 + d_0)}{2}\beta_0 + \frac{(a_1 - d_1)}{2}\beta_1 + \frac{(a_2 - d_2)}{2}\beta_2 + \frac{(a_3 - d_3)}{2}\beta_3 + \frac{(b_0 - c_0)}{2}\beta_4 + \frac{(b_1 + c_1)}{2}\beta_5 + \frac{(b_2 + c_2)}{2}\beta_6 + \frac{(b_3 + c_3)}{2}\beta_7 \\ &+ \frac{(a_1 + d_1)}{2}\beta_8 - \frac{(a_0 - d_0)}{2}\beta_9 + \frac{(a_3 + d_3)}{2}\beta_A - \frac{(a_2 + d_2)}{2}\beta_B + \frac{(b_1 - c_1)}{2}\beta_C - \frac{(b_0 + c_0)}{2}\beta_D + \frac{(b_3 - c_3)}{2}\beta_E - \frac{(b_2 - c_2)}{2}\beta_F \end{aligned} \tag{145}$$

$$= z_0 \times e_0 \;+\; z_1 \times e_1 \;+\; z_2 \times e_2 \;+\; z_3 \times e_3 \;+\; z_4 \times e_4 \;+\; z_5 \times e_5 \;+\; z_6 \times e_6 \;+\; z_7 \times e_7 \,, \qquad z_j \in \mathbb{C} \tag{146}$$

where,

$$\begin{aligned} z_0 &= \frac{1}{2}(a_0 + d_0) + \frac{1}{2}(a_1 + d_1)\boldsymbol{i}, \;\; z_1 = \frac{1}{2}(a_1 - d_1) - \frac{1}{2}(a_0 - d_0)\boldsymbol{i}, \;\; z_2 = \frac{1}{2}(a_2 - d_2) + \frac{1}{2}(a_3 + d_3)\boldsymbol{i}, \;\; z_3 = \frac{1}{2}(a_3 - d_3) - \frac{1}{2}(a_2 + d_2)\boldsymbol{i} \\ z_4 &= \frac{1}{2}(b_0 - c_0) + \frac{1}{2}(b_1 - c_1)\boldsymbol{i}, \;\; z_5 = \frac{1}{2}(b_1 + c_1) - \frac{1}{2}(b_0 + c_0)\boldsymbol{i}, \;\; z_6 = \frac{1}{2}(b_2 + c_2) + \frac{1}{2}(b_3 - c_3)\boldsymbol{i}, \;\; z_7 = \frac{1}{2}(b_3 + c_3) - \frac{1}{2}(b_2 - c_2)\boldsymbol{i} \end{aligned}$$

While this number can also be written, $h = \sum z_u \times e_u$, with complex number coefficients on the 8-dimensional octonion basis, (146), this is not the usual "bi-octonion," because the complex coefficients do not commute with all the octonion basis elements here. The product between the complex coefficients and the octonion basis makes use of the special non-associative twisted $\times$ product. So, if anything, these restricted quatro-quaternion numbers might be called, alternatively, the "twisted bi-octonions." The ordinary "bi-octonions" are represented by the tensor product $\mathbb{C} \otimes \mathbb{O}$, which we've just seen has a different structure from our twisted version, $\mathbb{C} \times \mathbb{O}$. Nevertheless, although our non-associative matrix algebra $M_{[\times]}(2, \mathbb{H})$ is not a tensor product, it *is* represented by the product of the two algebras, so we call this $\mathbb{C} \times \mathbb{O}$ construction, simply, a "product algebra." We can then write, $\mathbb{C} \times \mathbb{O} \cong M_{[\times]}(2, \mathbb{H})$.

`Conjecture:` $\mathbb{H} \times \mathbb{O} \cong M_{[\times]}(4, \mathbb{C})$. i.e. the product algebra of the quaternion algebra with the octonion algebra is isomorphic to the non-associative 4×4 matrix algebra over the complex numbers defined by the *derived twisted product* $\times$ given in (115).

—this conjecture turned out to be false, there being no quaternion triples found outside the octonion basis.

All the triples that have anti-commuting pairs form another type of four dimensional hypercomplex number, which we identified in our previous Hexpe paper [PJ2] [2] and labeled there "alternating complex numbers." While there are triples, $\boldsymbol{ijk}$, where all pairs anti-commute, they do not follow the "cyclical" pattern, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j} \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},$, characteristic of quaternions, instead they follow, $\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = -\boldsymbol{j} \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i},$, which is an "alternating" pattern, with a, $\boldsymbol{ki} = -\boldsymbol{j}$, left-hand turn, alternating with two right-hand turns, $\boldsymbol{ij} = +\boldsymbol{k}$ and $\boldsymbol{jk} = +\boldsymbol{i}$. Moreover, while, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$, for quaternions, these numbers have squares alternating between $+1$ and -1, with, $\boldsymbol{i}^2 = -\boldsymbol{j}^2 = \boldsymbol{k}^2 = +1$. Since, the *alternating complex numbers* seem to play a prominant role here, we shall introduce an official symbol[3], $\ddot{\mathbb{H}}$, for these numbers. The eight basis units, $\{\pm 1, \pm\boldsymbol{i}, \pm\boldsymbol{j}, \pm\boldsymbol{k}\}$, for $\ddot{\mathbb{H}}$, form the group D_4, the fourth dihedral group, which is the group of symmetries of the 2-space square. This is one of the five groups of order eight, with the set of basis quaternions, $Q \equiv \{\pm 1, \pm i, \pm j, \pm k\} \subset \mathbb{H}$, being another—these two groups, Q and D_4, are the only "non-abelian" groups among the five, the other three are all abelian. We then replace our conjecture with the following alternative.

`Conjecture:` $\ddot{\mathbb{H}} \times \mathbb{O} \cong M_{[\times]}(4, \mathbb{C})$. i.e. the product algebra of the alternating complex algebra with the octonion algebra is isomorphic to the non-associative 4×4 matrix algebra over the complex numbers defined by the *derived twisted product* $\times$ given in (115).

`Proof:`
Let an octonion be, $o = o_u e_u$, with basis, $e_u,\ u = 0, 1, 2, ..., 7$, and alternating complex number, $\alpha = w + \boldsymbol{i}x + \boldsymbol{j}y + \boldsymbol{k}z$. First we erect a basis, using the 4×4 matrix representation (116) for the octonion, then pick an independent $\boldsymbol{ijk}$.

$$o = \begin{pmatrix} a & -b^* & -c^* & -d^* \\ b & a^* & d & -c \\ c & -d^* & a^* & b^* \\ d & c^* & -b & a \end{pmatrix} = \begin{pmatrix} a_0 + ia_1 & -b_0 + ib_1 & -c_0 + ic_1 & -d_0 + id_1 \\ b_0 + ib_1 & a_0 - ia_1 & d_0 + id_1 & -c_0 - ic_1 \\ c_0 + ic_1 & -d_0 + id_1 & a_0 - ia_1 & b_0 - ib_1 \\ d_0 + id_1 & c_0 - ic_1 & -b_0 - ib_1 & a_0 + ia_1 \end{pmatrix} \qquad a, b, c, d \in \mathbb{C} \tag{147}$$

$$= a_0 \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} + a_1 \begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & +i \end{pmatrix} + b_0 \begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \end{pmatrix} + b_1 \begin{pmatrix} 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} \tag{148}$$

$$+ c_0 \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} + d_0 \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} + d_1 \begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \\ 0 & +i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix}$$

$$= a_0 e_0 + a_1 e_1 + b_0 e_2 + b_1 e_3 + c_0 e_4 + c_1 e_5 + d_0 e_6 + d_1 e_7 \tag{149}$$

$$\alpha = w \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} + x \begin{pmatrix} 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} + y \begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} + z \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix} \tag{150}$$

$$= w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad w, x, y, z \in \mathbb{R},\ \alpha \in \ddot{\mathbb{H}} \tag{151}$$

This octonion basis has four real 4×4 matrices, and four imaginary matrices. An independent $\boldsymbol{ijk}$ triple can be found by selecting two octonion basis matrices, modifying the $\pm$ signs on the entries to obtain independent matrices, then checking that their product is also independent of the original eight, and that together with their product they form a triple with the right properties defining an alternating complex number. One such triple is presented in (150). It is easily verified that this triple is associative under the twisted $\times$ product, i.e. $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$, for all assignments of $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ to the variables, $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$, despite the otherwise non-associative nature of the $\times$ product. So, the $\pm$ unit basis elements of the alternating complex number do actually form a non-abelian "group" under the twisted × product. This means that we have these associative subalgebras within the $M_{[\times]}(4,\mathbb{C})$ non-associative algebra, similar to the associative quaternion subalgebras that exist within that part formed from the octonion algebra.

$\times$	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
1	1	$\boldsymbol{i}$	$\boldsymbol{j}$	$\boldsymbol{k}$
$\boldsymbol{i}$	$\boldsymbol{i}$	1	$\boldsymbol{k}$	$\boldsymbol{j}$
$\boldsymbol{j}$	$\boldsymbol{j}$	$-\boldsymbol{k}$	-1	$\boldsymbol{i}$
$\boldsymbol{k}$	$\boldsymbol{k}$	$-\boldsymbol{j}$	$-\boldsymbol{i}$	1

$\ddot{\mathbb{H}}$ — ALTERNATING COMPLEX NUMBER PRODUCT TABLE FOR THE GROUP D_4

VS

$\times$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	$-j$	$-k$	-1	i
k	k	j	$-i$	-1

$\mathbb{H}$ — HAMILTON'S QUATERNIONS PRODUCT TABLE FOR THE GROUP Q

(152)

On multiplying the $\{1,\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}\}$ basis elements of $(150-151)$ we can confirm they produce the product table shown in (152) for the alternating complex number. This can be compared to the corresponding non-abelian 4-dim hypercomplex number product table for Hamilton's Quaternions shown on the right. Using the twisted × product to multiply the eight octonion basis matrices $(148-149)$, $e_u,\ u=0,1,2,\ldots,7$, we also obtain the following table;

$\times$	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_0	e_0	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	$-e_0$	$-e_3$	e_2	$-e_5$	e_4	e_7	$-e_6$
e_2	e_2	e_3	$-e_0$	$-e_1$	$-e_6$	$-e_7$	e_4	e_5
e_3	e_3	$-e_2$	e_1	$-e_0$	$-e_7$	e_6	$-e_5$	e_4
e_4	e_4	e_5	e_6	e_7	$-e_0$	$-e_1$	$-e_2$	$-e_3$
e_5	e_5	$-e_4$	e_7	$-e_6$	e_1	$-e_0$	e_3	$-e_2$
e_6	e_6	$-e_7$	$-e_4$	e_5	e_2	$-e_3$	$-e_0$	e_1
E_7	e_7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	$-e_0$

→

$\times$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	$\overline{0}$	$\overline{3}$	2	$\overline{5}$	4	7	$\overline{6}$
2	2	3	$\overline{0}$	$\overline{1}$	$\overline{6}$	$\overline{7}$	4	5
3	3	$\overline{2}$	1	$\overline{0}$	$\overline{7}$	6	$\overline{5}$	4
4	4	5	6	7	$\overline{0}$	$\overline{1}$	$\overline{2}$	$\overline{3}$
5	5	$\overline{4}$	7	$\overline{6}$	1	$\overline{0}$	3	$\overline{2}$
6	6	$\overline{7}$	$\overline{4}$	5	2	$\overline{3}$	$\overline{0}$	1
7	7	6	$\overline{5}$	$\overline{4}$	3	2	$\overline{1}$	$\overline{0}$

(153)

OCTONION PRODUCT TABLE IN THE $M_{[\times]}(4,\mathbb{C})$ REPRESENTATION. & COMPACT FORM.

At first glance, table (153) for the $M_{[\times]}(4,\mathbb{C})$ representation, and table (121) for the $M_{[\times]}(2,\mathbb{H})$ representation, look different. However, a simple re-arrangement shows they are the same table. If we swap labels, e_2 and e_3, and swap labels, e_6 and e_7, in (153), we obtain table (121). Thus, our match is obtained by the corresponding label assignments;

$$
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
1 & \boldsymbol{i} & \boldsymbol{j} & (\boldsymbol{ij}) & \boldsymbol{k} & (\boldsymbol{ki}) & (\boldsymbol{kj}) & (\boldsymbol{k}(\boldsymbol{ij})) \\
e_0 & e_1 & \boldsymbol{e_3} & \boldsymbol{e_2} & e_4 & e_5 & \boldsymbol{e_7} & \boldsymbol{e_6}
\end{array} \tag{154}
$$

The basis elements that require swapping all involve that same $\boldsymbol{j}$ from the three generator $\boldsymbol{ijk}$ octonion table (126), on which we previously imposed the special "mystery twist" (123) to get things to match before without re-arranging table elements. Now, we essentially have to undo that twist to see the equivalence between the tables. We could have avoided all this re-arranging by simply taking the ijk quaternion basis introduced in (119) to be "left-handed," at the outset, when computing table (121). Then we wouldn't need to employ the (123) reverse operator, to get things to match before, and wouldn't have to undo that twist to see the equivalence here again. But, this way we highlight one of the characteristics of octonions, that the 7 quaternion triples may appear in either R-H or L-H format, and thus create different looking tables that are nevertheless "isomorphically" the same. Now that we have an octonion basis, and linearly independent alternating complex number triple, $\boldsymbol{ijk}$, we combine them to expand the basis.

$$
\begin{array}{cccccccc}
\boldsymbol{i}\times e_0 & \boldsymbol{i}\times e_1 & \boldsymbol{i}\times e_2 & \boldsymbol{i}\times e_3 & \boldsymbol{i}\times e_4 & \boldsymbol{i}\times e_5 & \boldsymbol{i}\times e_6 & \boldsymbol{i}\times e_7 \\
\begin{pmatrix} 0 & +i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} &
\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} &
\begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & +i & 0 \\ 0 & 0 & 0 & +i \end{pmatrix} &
\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \\ 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & +i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
\boldsymbol{j}\times e_0 & \boldsymbol{j}\times e_1 & \boldsymbol{j}\times e_2 & \boldsymbol{j}\times e_3 & \boldsymbol{j}\times e_4 & \boldsymbol{j}\times e_5 & \boldsymbol{j}\times e_6 & \boldsymbol{j}\times e_7 \\
\begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & +i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} &
\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix} &
\begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix} &
\begin{pmatrix} 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & +i \\ +i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & +i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} \\
\boldsymbol{k}\times e_0 & \boldsymbol{k}\times e_1 & \boldsymbol{k}\times e_2 & \boldsymbol{k}\times e_3 & \boldsymbol{k}\times e_4 & \boldsymbol{k}\times e_5 & \boldsymbol{k}\times e_6 & \boldsymbol{k}\times e_7 \\
\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & +1 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & +i & 0 \end{pmatrix} &
\begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} &
\begin{pmatrix} +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & 0 & +i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ +i & 0 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \\ +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} &
\begin{pmatrix} 0 & 0 & +i & 0 \\ 0 & 0 & 0 & +i \\ +i & 0 & 0 & 0 \\ 0 & +i & 0 & 0 \end{pmatrix}
\end{array} \tag{155}
$$

The 24 matrices in (155), together with the previous 8 octonion basis matrices in (148), form a complete set of 32 linearly independent basis matrices that represent the $M_{[\times]}(4,\mathbb{C})$ algebra. To see that they are linearly independent matrices, notice that they all fall into groups of 4 where the matrix entries are in the same positions, just with differing signs; e.g. e_1, $\boldsymbol{i}\times e_2$, $\boldsymbol{j}\times e_0$, $\boldsymbol{k}\times e_3$, all have the complex i unit along the main diagonal. If these four were linearly dependent, there would then exist real parameters, λ_u , for which $\lambda_0 e_1 + \lambda_1 \boldsymbol{i}\times e_2 + \lambda_2 \boldsymbol{j}\times e_0 + \lambda_3 \boldsymbol{k}\times e_3 = 0$, leading to the system of four simultaneous equations,

$$\begin{aligned} +\lambda_0+\lambda_1+\lambda_2+\lambda_3&=0\\ -\lambda_0+\lambda_1-\lambda_2+\lambda_3&=0\\ -\lambda_0+\lambda_1+\lambda_2-\lambda_3&=0\\ +\lambda_0+\lambda_1-\lambda_2-\lambda_3&=0 \end{aligned} \tag{156}$$

which, however, only have solution, $\lambda_0=\lambda_1=\lambda_2=\lambda_3=0$. So, all the 32 matrices in this basis set for $M_{[\times]}(4,\mathbb{C})$ are linearly independent. Hence, $\ddot{\mathbb{H}}\times\mathbb{O}\cong M_{[\times]}(4,\mathbb{C})$, and the non-associative 4×4 matrix algebra over the complex numbers is isomorphic to the "product algebra" of the alternating complex algebra and the octonion algebra.

Any number $h\in M_{[\times]}(4,\mathbb{C})$ can be represented by this 32-dim $\ddot{\mathbb{H}}\times\mathbb{O}$ basis,

$$h = \sum_{u=0}^{7}\lambda_{u,0}e_u + \sum_{u=0}^{7}\lambda_{u,1}\boldsymbol{i}\times e_u + \sum_{u=0}^{7}\lambda_{u,2}\boldsymbol{j}\times e_u + \sum_{u=0}^{7}\lambda_{u,3}\boldsymbol{k}\times e_u \tag{157}$$

$$= \sum_{u=0}^{7}\alpha_u\times e_u \tag{158}$$

where,

$$\alpha_u = \lambda_{u,0}+\lambda_{u,1}\boldsymbol{i}+\lambda_{u,2}\boldsymbol{j}+\lambda_{u,3}\boldsymbol{k}\ , \qquad \lambda_{u,s}\in\mathbb{R},\ \alpha_u\in\ddot{\mathbb{H}} \tag{159}$$

But, notice again, that the product algebra, $\ddot{\mathbb{H}}\times\mathbb{O}$, is different from the corresponding tensor product, $\ddot{\mathbb{H}}\otimes\mathbb{O}$. The next natural question to ponder, is whether there is any relationship between between $\ddot{\mathbb{H}}\times\mathbb{O}$ and $\mathbb{H}\otimes\mathbb{O}$. However, the ijk imaginary units in quaternions have an interchangeable symmetry that would suggest the tensor product show no special preference for $\boldsymbol{i},\boldsymbol{j}$ or $\boldsymbol{k}$. While, an asymmetry is easily seen in the units of $\ddot{\mathbb{H}}$. When we reverse the order of factors, these products in (155) fall into sets, of 2 and 6, that commute and anti-commute.

$$\begin{array}{lllll} \boldsymbol{i}: & e_u\times\boldsymbol{i}=\boldsymbol{i}\times e_u, & u=0,2. & e_u\times\boldsymbol{i}=-\boldsymbol{i}\times e_u, & u=1,3,4,5,6,7.\\ \boldsymbol{j}: & e_u\times\boldsymbol{j}=\boldsymbol{j}\times e_u, & u=0,1,4,5,6,7. & e_u\times\boldsymbol{j}=-\boldsymbol{j}\times e_u, & u=2,3.\\ \boldsymbol{k}: & e_u\times\boldsymbol{k}=\boldsymbol{k}\times e_u, & u=0,3,4,5,6,7. & e_u\times\boldsymbol{k}=-\boldsymbol{k}\times e_u, & u=1,2. \end{array} \tag{160}$$

But, the pattern "alternates" among the $\boldsymbol{ijk}$ units, with $\boldsymbol{i}$ commuting 2 and anti-commuting 6, while $\boldsymbol{j}$ and $\boldsymbol{k}$ reverse this enumeration, commuting 6 and anti-commuting 2, instead. Thus, this singles out one unit, $\boldsymbol{i}$, as special (although, initially one might think $\boldsymbol{j}$ should be the special unit), and indicates $\ddot{\mathbb{H}}\times\mathbb{O}$ is also different from $\mathbb{H}\otimes\mathbb{O}$.

HEXPE NUMBERS. The 32 basis matrices in this set form two distinct subsets, of 16 real 4×4 matrices, and 16 purely imaginary 4×4 matrices. The set of 4×4 real matrices happen to be identical to that which forms the basis set for the `hexpe numbers`, and is a real matrix "$\times$" subalgebra of the $M_{[\times]}(4,\mathbb{C})$, which we shall label $M_{[\times]}(4,\mathbb{R})$.

E	I_R	J_R	K_R	I_L	J_L	K_L	I_M	J_M	K_M	I_A	J_A	K_A	I_Z	J_Z	K_Z	(161)
$e0$	$\boldsymbol{j}\times e_3$	$\boldsymbol{k}\times e_6$	$\boldsymbol{k}\times e_4$	e_2	e_4	e_6	$\boldsymbol{j}\times e_1$	$-\boldsymbol{k}\times e_2$	$\boldsymbol{i}\times e_3$	$\boldsymbol{i}\times e_1$	$\boldsymbol{i}\times e_7$	$\boldsymbol{j}\times e_7$	$-\boldsymbol{k}\times e_0$	$\boldsymbol{j}\times e_5$	$\boldsymbol{i}\times e_5$	

The Hexpentaquaternion basis, $\mathbb{X}_b = \pm\{E,I_R,J_R,K_R,I_L,J_L,K_L,I_M,J_M,K_M,I_A,J_A,K_A,I_Z,J_Z,K_Z\}$, is closed under the twisted $\times$ product. So, we may now extend this associative algebra by including the non-associative product. The 16-dimensional "dual-product" Hexpentaquaternion algebra, $(\mathbb{X}_n,\cdot,\times)$, is then a subalgebra of the $M_{[\cdot,\times]}(4,\mathbb{C})$, and we may alternatively refer to it as $M_{[\cdot,\times]}(4,\mathbb{R})$. The correspondence, between the previously defined `hexpe` basis elements—[PJ2] [2] (page 59, Table T.1)—and the 16 real basis matrices given here, is shown in table (161).

The remaining set of 16 imaginary 4×4 matrices can all be re-written iM, where i is the ordinary complex imaginary unit, and M is a real 4×4 matrix, with $M\in\mathbb{X}_b$, once again.

THE TWISTED $\times$ PRODUCT INVERSE FOR $M_{[\times]}(4,\mathbb{C})$

We have seen how to construct twisted product inverse matrices for $M_{[\times]}(2,\mathbb{H})$. But, now we have generalized the algebra in expanding to 4×4, and need to re-consider the inverse again. Fortunately, this time, all our matrix entries commute. But, we do still need to take into account the effects of *percolation*. Consider the following 4×4 equation,

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad a_{ij}, b_{ij} \in \mathbb{C} \tag{162}$$

where, $[a_{ij}]$, is the known, and, $[b_{ij}]$, the unknown. We can expand this matrix expression and re-arrange the resulting equations into an equivalent form expressible by the standard matrix algebra. But, this time we need 16-dim arrays. It helps to see the results of the $\times$ product side by side with that for the standard $\cdot$ product, when both are written out and re-expressed in the 16-dim format. The following two equations show this comparison.

$$\begin{pmatrix}
\begin{matrix} a_{00} & a_{01} & a_{02} & \mathbf{a_{03}} \\ a_{10} & a_{11} & \mathbf{a_{12}} & \mathbf{a_{13}} \\ a_{20} & \mathbf{a_{21}} & a_{22} & a_{23} \\ \mathbf{a_{30}} & \mathbf{a_{31}} & a_{32} & a_{33} \end{matrix} & 0 & 0 & 0 \\
0 & \begin{matrix} a_{00} & a_{01} & \mathbf{a_{02}} & \mathbf{a_{03}} \\ a_{10} & a_{11} & \mathbf{a_{12}} & a_{13} \\ \mathbf{a_{20}} & \mathbf{a_{21}} & a_{22} & a_{23} \\ \mathbf{a_{30}} & a_{31} & a_{32} & a_{33} \end{matrix} & 0 & 0 \\
0 & 0 & \begin{matrix} a_{00} & \mathbf{a_{01}} & a_{02} & a_{03} \\ \mathbf{a_{10}} & \mathbf{a_{11}} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & \mathbf{a_{23}} \\ a_{30} & a_{31} & \mathbf{a_{32}} & \mathbf{a_{33}} \end{matrix} & 0 \\
0 & 0 & 0 & \begin{matrix} \mathbf{a_{00}} & \mathbf{a_{01}} & a_{02} & a_{03} \\ \mathbf{a_{10}} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & \mathbf{a_{22}} & \mathbf{a_{23}} \\ a_{30} & a_{31} & \mathbf{a_{32}} & a_{33} \end{matrix}
\end{pmatrix}
\begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \\ b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \\ b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \\ b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix}
=
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{163}$$

The effects of percolation are clearer when viewed in terms of the standard matrix transformation equations. Above we have the standard associative $\cdot$ product for $M(4,\mathbb{C})$. Below, the non-associative twisted $\times$ product for $M_{[\times]}(4,\mathbb{C})$. The percolating entities are marked in **boldface**. The diagonal form is modified by the inclusion of satellite matrices.

$$\begin{pmatrix}
\begin{matrix} a_{00} & a_{01} & a_{02} & \cdot \\ a_{10} & a_{11} & \cdot & \mathbf{a_{02}} \\ a_{20} & \cdot & a_{22} & a_{23} \\ \cdot & \mathbf{a_{20}} & a_{32} & a_{33} \end{matrix} & \begin{matrix} \cdot & \cdot & \mathbf{a_{12}} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{a_{12}} \\ \mathbf{a_{30}} & \cdot & \cdot & \cdot \\ \cdot & \mathbf{a_{30}} & \cdot & \cdot \end{matrix} & 0 & 0 \\
\begin{matrix} \cdot & \cdot & \mathbf{a_{03}} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{a_{03}} \\ \mathbf{a_{21}} & \cdot & \cdot & \cdot \\ \cdot & \mathbf{a_{21}} & \cdot & \cdot \end{matrix} & \begin{matrix} a_{00} & a_{01} & \mathbf{a_{13}} & \cdot \\ a_{10} & a_{11} & \cdot & a_{13} \\ \mathbf{a_{31}} & \cdot & a_{22} & a_{23} \\ \cdot & a_{31} & a_{32} & a_{33} \end{matrix} & 0 & 0 \\
0 & 0 & \begin{matrix} a_{00} & \cdot & a_{02} & a_{03} \\ \cdot & \mathbf{a_{00}} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & \cdot \\ a_{30} & a_{31} & \cdot & \mathbf{a_{22}} \end{matrix} & \begin{matrix} \mathbf{a_{10}} & \cdot & \cdot & \cdot \\ \cdot & \mathbf{a_{10}} & \cdot & \cdot \\ \cdot & \cdot & \mathbf{a_{32}} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{a_{32}} \end{matrix} \\
0 & 0 & \begin{matrix} \mathbf{a_{01}} & \cdot & \cdot & \cdot \\ \cdot & \mathbf{a_{01}} & \cdot & \cdot \\ \cdot & \cdot & \mathbf{a_{23}} & \cdot \\ \cdot & \cdot & \cdot & \mathbf{a_{23}} \end{matrix} & \begin{matrix} \mathbf{a_{11}} & \cdot & a_{02} & a_{03} \\ \cdot & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & \mathbf{a_{33}} & \cdot \\ a_{30} & a_{31} & \cdot & a_{33} \end{matrix}
\end{pmatrix}
\begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \\ b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \\ b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \\ b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix}
=
\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{164}$$

[In the square array, the dots " $\cdot$ " are isolated zeros, and the 0's are entire 4×4 matrices of zeros].

The main feature of note, is that, whereas, in the standard $\cdot$ product above, the 4×1 column vectors separate, each effectively producing their own independent matrix equation, in the non-associative $\times$ product below, two 4×1 column vectors are coupled together into linked pairs, that do not separate, and are not generally separable.

The first two columns in the $[b_{ij}]$ matrix, therefore, form the following system of coupled 4-dim matrix equations, where the general transformation $[a_{ij}]$ matrix is mangled by the percolating action, and split apart into two unequally distributed matrices. The sparse matrix we refer to as the "satellite matrix," the denser simply a "percolated matrix."

$$\begin{pmatrix} a_{00} & a_{01} & a_{02} & \cdot \\ a_{10} & a_{11} & \cdot & \boldsymbol{a_{02}} \\ a_{20} & \cdot & a_{22} & a_{23} \\ \cdot & \boldsymbol{a_{20}} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} + \begin{pmatrix} \cdot & \cdot & \boldsymbol{a_{12}} & \cdot \\ \cdot & \cdot & \cdot & \boldsymbol{a_{12}} \\ \boldsymbol{a_{30}} & \cdot & \cdot & \cdot \\ \cdot & \boldsymbol{a_{30}} & \cdot & \cdot \end{pmatrix} \begin{pmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{165}$$

$$\begin{pmatrix} \cdot & \cdot & \boldsymbol{a_{03}} & \cdot \\ \cdot & \cdot & \cdot & \boldsymbol{a_{03}} \\ \boldsymbol{a_{21}} & \cdot & \cdot & \cdot \\ \cdot & \boldsymbol{a_{21}} & \cdot & \cdot \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} + \begin{pmatrix} a_{00} & a_{01} & \boldsymbol{a_{13}} & \cdot \\ a_{10} & a_{11} & \cdot & a_{13} \\ \boldsymbol{a_{31}} & \cdot & a_{22} & a_{23} \\ \cdot & a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{166}$$

The satellite matrices remain associated with the same pair of percolated matrices with which they share a common origin. The four cross diagonal elements, in this case, are ripped out of the original general matrix, and used to populate the satellites. The remaining percolated matrix bodies also undergo other minor modifications shown in **boldface**. But, if we associate each satellite matrix with its percolated partner that shares the same horizontal positioning in the (164) square array, we then have four percolated, A_1, A_2, A_2, A_4, and their four corresponding satellites, S_1, S_2, S_3, S_4, making up the system of equations. The first paired system of equations can then be written,

$$\begin{aligned} A_1 b_1 + S_1 b_2 &= c_1 & \qquad b_1 &= (S_1^{-1} A_1 - A_2^{-1} S_2)^{-1} (S_1^{-1} c_1 - A_2^{-1} c_2) \\ S_2 b_1 + A_2 b_2 &= c_2 & \qquad b_2 &= (A_1^{-1} S_1 - S_2^{-1} A_2)^{-1} (A_1^{-1} c_1 - S_2^{-1} c_2) \end{aligned} \tag{167}$$

where, b_1, b_2, b_3, b_4, are the corresponding 4×1 column vectors from the $[b_{ij}]$ matrix, and c_1, c_2, c_3, c_4, for this problem, are the related parts of the unit matrix. Since we're dealing with ordinary matrix algebra over complex numbers, the standard methods for inverting a matrix applies, and we can write the solution for the pair (b_1, b_2) in terms of the inverted matrices and their products as shown on the right in (167). These equations uncouple when the four entries in the satellite matrices vanish, which occurs when, $a_{12} = a_{21} = a_{03} = a_{30} = 0$, in the original general matrix. However, there's still an effect due to percolation owing to the other off-diagonal modifications.

$$\begin{pmatrix} a_{00} & \cdot & a_{02} & a_{03} \\ \cdot & \boldsymbol{a_{00}} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & \cdot \\ a_{30} & a_{31} & \cdot & \boldsymbol{a_{22}} \end{pmatrix} \begin{pmatrix} b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} + \begin{pmatrix} \boldsymbol{a_{10}} & \cdot & \cdot & \cdot \\ \cdot & \boldsymbol{a_{10}} & \cdot & \cdot \\ \cdot & \cdot & \boldsymbol{a_{32}} & \cdot \\ \cdot & \cdot & \cdot & \boldsymbol{a_{32}} \end{pmatrix} \begin{pmatrix} b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{168}$$

$$\begin{pmatrix} \boldsymbol{a_{01}} & \cdot & \cdot & \cdot \\ \cdot & \boldsymbol{a_{01}} & \cdot & \cdot \\ \cdot & \cdot & \boldsymbol{a_{23}} & \cdot \\ \cdot & \cdot & \cdot & \boldsymbol{a_{23}} \end{pmatrix} \begin{pmatrix} b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} + \begin{pmatrix} \boldsymbol{a_{11}} & \cdot & a_{02} & a_{03} \\ \cdot & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & \boldsymbol{a_{33}} & \cdot \\ a_{30} & a_{31} & \cdot & a_{33} \end{pmatrix} \begin{pmatrix} b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{169}$$

The remaining pair of column vectors, (b_2, b_3), can be determined in a similar manner. Note that the percolating action exhibits a different pattern here, which is, however, somewhat complementary to that in the first pair.

$$\begin{aligned} A_3 b_3 + S_3 b_4 &= c_3 & \qquad b_3 &= (S_3^{-1} A_3 - A_4^{-1} S_4)^{-1} (S_3^{-1} c_3 - A_4^{-1} c_4) \\ S_4 b_3 + A_4 b_4 &= c_4 & \qquad b_4 &= (A_3^{-1} S_3 - S_4^{-1} A_4)^{-1} (A_3^{-1} c_3 - S_4^{-1} c_4) \end{aligned} \tag{170}$$

When all of the matrix entries that populate satellites vanish, in the original general matrix, we obtain a special matrix (171) that looks somewhat like spaghetti. In this case, all the four column 4×1 vectors from the $[b_{ij}]$ decouple, and the system of equations partitions into four individual matrix problems just like in the standard matrix algebra.

$$\underset{\text{GENERAL MATRIX}}{\begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}} \quad \longrightarrow \quad \underset{\text{SPAGHETTI MATRIX}}{\begin{pmatrix} a_{00} & \cdot & a_{02} & \cdot \\ \cdot & a_{11} & \cdot & a_{13} \\ a_{20} & \cdot & a_{22} & \cdot \\ \cdot & a_{31} & \cdot & a_{33} \end{pmatrix}} \tag{171}$$

We can then ask sensible questions like "What are the non-associative twisted product eigenvalues and eigenvectors for a spaghetti matrix?", because, 4×1 column vectors can once again be treated as isolated objects that undergo individual transformations. But, notice that, if the original matrix has spaghetti form, the non-associative product still produces four different percolated spaghetti matrices, A_1, A_2, A_3, A_4, that each have their own eigen-parameters.

$$\begin{pmatrix} a_{00} & \cdot & a_{02} & \cdot \\ \cdot & a_{11} & \cdot & a_{13} \\ a_{20} & \cdot & a_{22} & \cdot \\ \cdot & a_{31} & \cdot & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{00} & b_{01} & b_{02} & b_{03} \\ b_{10} & b_{11} & b_{12} & b_{13} \\ b_{20} & b_{21} & b_{22} & b_{23} \\ b_{30} & b_{31} & b_{32} & b_{33} \end{pmatrix} \quad \rightarrow \tag{172}$$

$$\begin{pmatrix} a_{00} & \cdot & a_{02} & \cdot \\ \cdot & a_{11} & \cdot & \mathbf{a_{02}} \\ a_{20} & \cdot & a_{22} & \cdot \\ \cdot & \mathbf{a_{20}} & \cdot & a_{33} \end{pmatrix} \begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \end{pmatrix} \quad \begin{pmatrix} a_{00} & \cdot & \mathbf{a_{13}} & \cdot \\ \cdot & a_{11} & \cdot & a_{13} \\ \mathbf{a_{31}} & \cdot & a_{22} & \cdot \\ \cdot & a_{31} & \cdot & a_{33} \end{pmatrix} \begin{pmatrix} b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} \quad \begin{pmatrix} a_{00} & \cdot & a_{02} & \cdot \\ \cdot & \mathbf{a_{00}} & \cdot & a_{13} \\ a_{20} & \cdot & a_{22} & \cdot \\ \cdot & a_{31} & \cdot & \mathbf{a_{22}} \end{pmatrix} \begin{pmatrix} b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} \quad \begin{pmatrix} \mathbf{a_{11}} & \cdot & a_{02} & \cdot \\ \cdot & a_{11} & \cdot & a_{13} \\ a_{20} & \cdot & \mathbf{a_{33}} & \cdot \\ \cdot & a_{31} & \cdot & a_{33} \end{pmatrix} \begin{pmatrix} b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix}$$

$$A_1 b_1 = \lambda_1 b_1 \ , \qquad A_2 b_2 = \lambda_2 b_2 \ , \qquad A_3 b_3 = \lambda_3 b_3 \ , \qquad A_4 b_4 = \lambda_4 b_4 \ . \tag{173}$$

$$\begin{array}{ll} (\lambda_1): \ (a_{22} - \lambda_1)(a_{00} - \lambda_1) = (a_{33} - \lambda_1)(a_{11} - \lambda_1) = a_{02}a_{20} & (\lambda_3): \ (a_{22} - \lambda_3)(a_{00} - \lambda_3) = a_{02}a_{20} = a_{13}a_{31} \\ (\lambda_2): \ (a_{22} - \lambda_2)(a_{00} - \lambda_2) = (a_{33} - \lambda_2)(a_{11} - \lambda_2) = a_{13}a_{31} & (\lambda_4): \ (a_{33} - \lambda_4)(a_{11} - \lambda_4) = a_{02}a_{20} = a_{13}a_{31} \end{array} \tag{174}$$

The four eigenvalue parameters, $\lambda_j, \ j = 1, 2, 3, 4$, must each satisfy a pair of quadratic equations, shown in (174). All eigenvalues exist and are shared in common whenever $\exists \, \lambda\!: \ (a_{00} - \lambda)(a_{22} - \lambda) = (a_{11} - \lambda)(a_{33} - \lambda) = a_{02}a_{20} = a_{13}a_{31}$. We can then speak of "the" eignevalue of the original spaghetti matrix, otherwise eigenvalues are specific to the percolated spaghetti matrices derived from the original transformation matrix. But, transformations defined by spaghetti type matrices act on individual column vectors somewhat similar to the situation in ordinary matrix algebra.

In general, however, the non-associative matrix algebra is quite different from the usual associative matrix algebra. Collections of vectors link up and transform as a group, not as individualised objects. For this reason, one considers the action of a square matrix on another square matrix, rather than the more familiar theme of a square matrix acting on a single isolated column vector. If we consider the 4×1 column vector to represent the position of a point in 4-dimensional complex space, then the twisted product links pairs of points together, and the non-associative matrix algebra describes the transformations of those linked pairs rather than the usual point transformations found in the more familiar applications of standard linear algebra. When we expand this 4×4 matrix algebra to 8×8, the linked pair of 4×1 column vectors doubles again, and we now have 4 column vectors that link up and transform as a group together. Since, however, each of these new column vectors has dimension 8×1, we can fit the descriptive parameters for two points of a 4-dimensional real space in each column. So, effectively, 8 points link up and transform together. If we identify the 8 verticies of a 3-dimensional cube to be those 8 points, then the non-associative algebra $M_{[\times]}(8, \mathbb{R})$ describes the linked transformation of entire three-dimensional objects, or the cubic lattice of space.

If we enumerate the real-valued parameters in each algebra, we see that the twisted product links 2 spacetime points in $M_{[\times]}(2, \mathbb{H})$, using one column vector, links 4 spacetime points in $M_{[\times]}(4, \mathbb{C})$, using two coupled column vectors, and links 8 spacetime points in $M_{[\times]}(8, \mathbb{R})$, using four coupled column vectors. Thus the non-associative algebra describes the transformation geometry of line segments, squares, and cubes, respectively, if one thinks of these 4-parameter point coordinates as event variables that identify the spatial location of select 3-space points at the same specified coordinate time value.

PHYSICAL APPLICATIONS. Although it is usual to often consider vectors as isolated objects, in order to dissect and study the phenomenal world, in reality many physical vectors are linked together. The classic example is the electromagnetic field. One can consider electric vectors, $\overrightarrow{\mathbf{E}}$, and magnetic vectors, $\overrightarrow{\mathbf{B}}$, in isolation. But, nature actually links them, regardless of how we choose to study these fields. In the classical field theory, the electric field has 3 components, $\overrightarrow{\mathbf{E}} = (E_x, E_y, E_z)$, and the magnetic field has 3 components, $\overrightarrow{\mathbf{B}} = (B_x, B_y, B_z)$. So, when one decides to recognise the fact that these fields couple together, and transform together, as a pair, $(\overrightarrow{\mathbf{E}}, \overrightarrow{\mathbf{B}})$, then one has to consider the issue of representation. The choice is, either create a double column vector with the six component parameters, and then describe transformations with 6×6 matrices, or dump the electromagnetic field parameters into a 3×3 matrix, using up only 6 of the 9 degrees of freedom, and consider transformations on that field-matrix instead. If we represent the electric and magnetic fields by quaternions, we now have two 4×1 column vectors that pair up, which matches the order of dimensions in our non-associative algebras, and so we consider this topic next.

MAXWELL'S EQUATIONS IN QUATERNIONS

$$\left[\frac{d}{dr} \,,\, B \right] = +\left\{ \frac{d}{dr} \,,\, E \right\} \qquad \left[\frac{d}{dr} \,,\, E \right] = -\left\{ \frac{d}{dr} \,,\, B \right\} \tag{175}$$

The quaternion electromagnetic theory, discussed in our previous paper on Maxwell's Equations [PJ1] [1], represents the electric and magnetic fields in terms of quaternion variables. The pair of fields (E, B) are thus practical examples of 4-parameter field vectors linked together by physical requirements, as shown in (175). The expressions that define the symmetric $\{d/dr, F\}$, and anti-symmetric $[d/dr, F]$, derivatives of a field, F, produce natural "twisted product" expressions when unfolded into their defining formulas.

$$\frac{d}{dr} \to B - B \leftarrow \frac{d}{dr} \;=\; \frac{d}{dr} \to E + E \leftarrow \frac{d}{dr} \;, \qquad \frac{d}{dr} \to (B+E) + (B-E) \leftarrow \frac{d}{dr} = 0$$

$$\frac{d}{dr} \to E - E \leftarrow \frac{d}{dr} \;=\; -\frac{d}{dr} \to B - B \leftarrow \frac{d}{dr} \;, \qquad -(B+E) \leftarrow \frac{d}{dr} + \frac{d}{dr} \to (B-E) = 0 \tag{176}$$

The equations on the left in (176) result from expanding the abbreviated forms in (175). They can then be re-arranged into the equations on the right, in terms of an alternative pair of quaternion fields, $(B+E)$ and $(B-E)$. This pair of equations can then be "doubled up" to obtain the two columns in a 2×2 quaternion matrix, which then allows us to write the electromagnetic equations as a matrix equation using the non-associative twisted $\times$ product.

Maxwell's Equations in $M_{[\times]}(2, \mathbb{H})$ representation.

$$\begin{pmatrix} \frac{d}{dr} \to (B+E) + (B-E) \leftarrow \frac{d}{dr} & (B-E) \leftarrow \frac{d}{dr} + \frac{d}{dr} \to (B+E) \\ -(B+E) \leftarrow \frac{d}{dr} + \frac{d}{dr} \to (B-E) & -\frac{d}{dr} \to (B-E) + (B+E) \leftarrow \frac{d}{dr} \end{pmatrix} = \begin{pmatrix} \frac{d}{dr} & \frac{d}{dr} \\ -\frac{d}{dr} & \frac{d}{dr} \end{pmatrix} \times \begin{pmatrix} B+E & B-E \\ B-E & B+E \end{pmatrix} = 0 \tag{177}$$

One way of writing Maxwell's Equations in the $M_{[\times]}(2, \mathbb{H})$ algebra is shown in (177). It is to be understood that the derivative operator, d/dr , always acts on the variable, regardless of whether it appears on the R.H.S or L.H.S of the variable, on expanding the matrix product. The natural coupling, of a pair of vector fields together, is accurately encapsulated in the relative reversing of factors characteristic of the twisted $\times$ product given by formal definition (2). So, the same modification to the matrix product definition, that enables representation of octonions and mimics the distinctive twist action intrinsic to the Cayley-Dickson process, also facilitates the expression of Maxwell's Electromagnetism by a simple matrix equation. This single matrix equation (177) can be re-written,

$$D \times \mathcal{B} = -D \times \mathcal{E} \qquad \Leftarrow \text{THE TWISTED } \times \text{ PRODUCT MAXWELL EQUATION(S)} \tag{178}$$

where,

$$D = \begin{pmatrix} d/dr & d/dr \\ -d/dr & d/dr \end{pmatrix} , \quad \mathcal{E} = \begin{pmatrix} +E & -E \\ -E & +E \end{pmatrix} , \quad \mathcal{B} = \begin{pmatrix} B & B \\ B & B \end{pmatrix} .$$

= or =,

$$D = \begin{pmatrix} d/dr & -d/dr \\ d/dr & d/dr \end{pmatrix} , \quad \mathcal{B} = \begin{pmatrix} +B & -B \\ -B & +B \end{pmatrix} , \quad \mathcal{E} = \begin{pmatrix} E & E \\ E & E \end{pmatrix} .$$

The signs in the matrix representations for the three parameters, $\{ D, \mathcal{E}, \mathcal{B} \}$, are not unique, but can be expressed in several alternative ways. Two such possibilities, that mirror each other, are shown in (178). The sign differences within the 2×2 matrix definitions for the $\mathbb{QQ}$-field parameters, $\mathcal{E}$ and $\mathcal{B}$, reflect the electromagnetic "duality," also variously called the "field exchange symmetry," or the "Heaviside Symmetry." It was first discovered by Oliver Heaviside, in the 1800s, that the classical 3-vector form of the homogeneous Maxwell's Equations remain invariant under a particular exchange of fields. If we replace $\vec{\mathbf{E}}$ with $\vec{\mathbf{B}}$ and $\vec{\mathbf{B}}$ with $-\vec{\mathbf{E}}$ we obtain the same equations again. This is on record as the very first symmetry ever identified in Maxwell's Equations. But, somehow, Heaviside did not notice, that if $q\colon (\vec{\mathbf{E}}, \vec{\mathbf{B}}) \mapsto (\vec{\mathbf{B}}, -\vec{\mathbf{E}})$ then $q^2\colon (\vec{\mathbf{E}}, \vec{\mathbf{B}}) \mapsto -(\vec{\mathbf{E}}, \vec{\mathbf{B}})$, so that, $q^2 = -1$, and existence of this "negative square" symmetry indicates the quaternionic formulation, which Maxwell had attempted, is perhaps much more "natural" to the basic structure of electromagnetism, than the "artificial" positive square vectors he introduced.

Heaviside complained that the quaternions possessed those "unnatural" negative squares, that shouldn't be part of physical theory, and explained that this was part of his motivation for replacing Hamilton's Quaternions with his (and Prof. Gibbs) vector formulation. In doing so, he was appealing to his prejudice for *the old familiar*, and was unable *to see the new* quaternionic implications in the very symmetry he himself had just discovered.

This symmetry is preserved in our quaternion re-formulation of Maxwell's Equations, as can be seen in (175). This time, exchanging the "quaternion fields," E with B and B with $-E$, we obtain the same quaternion homogeneous equations. When, therefore, we write the Maxwell's Equations as a single equation instead, i.e. $D \times \mathcal{B} = -D \times \mathcal{E}$, there are no longer 4 or 2 equations to exhibit the property, so the symmetry must be reflected in the structure of the field parameters, $\mathcal{E}$ and $\mathcal{B}$, themselves. If we keep the differential operator D fixed, and apply the field exchange to the internal entries in the 2×2 matrix representations for $\mathcal{E}$ and $\mathcal{B}$, we obtain the same equation again. So, the Heaviside Symmetry is once again preserved, even in the single equation $\mathbb{QQ}$ formulation given in (178).

Since quaternions are also 2×2 matrices over the complex numbers, we can now substitute the following (5) forms to expand this representation to the corresponding 4×4 matrix algebra over complex numbers; $z_1, z_2, E_1, E_2, B_1, B_2 \in \mathbb{C}$.

$$\frac{d}{dr} = \begin{pmatrix} \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ \frac{d}{dz_2} & \frac{d}{dz_1^*} \end{pmatrix} \qquad E = \begin{pmatrix} E_1 & -E_2^* \\ E_2 & E_1^* \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & -B_2^* \\ B_2 & B_1^* \end{pmatrix} \tag{179}$$

The natural pairing of vectors (E, B) is represented here in the 2×2 matrix by a column vector of the corresponding field pair $(B + E, B - E)$. This single 2×1 column vector now itself becomes a pair of 4×1 column vectors when we expand the $M_{[\times]}(2, \mathbb{H})$ matrix equation to $M_{[\times]}(4, \mathbb{C})$. So, two columns transform together in the 4×4 algebra, the linkage provided this time by the percolation induced by the twisted action of the $\times$ product in the higher algebra.

MAXWELL'S EQUATIONS IN $M_{[\times]}(4, \mathbb{C})$ REPRESENTATION:

$$\begin{pmatrix} \frac{d}{dz_1} & -\frac{d}{dz_2^*} & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ \frac{d}{dz_2} & \frac{d}{dz_1^*} & \frac{d}{dz_2} & \frac{d}{dz_1^*} \\ -\frac{d}{dz_1} & \frac{d}{dz_2^*} & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ -\frac{d}{dz_2} & -\frac{d}{dz_1^*} & \frac{d}{dz_2} & \frac{d}{dz_1^*} \end{pmatrix} \times \begin{pmatrix} B_1 + E_1 & -(B_2 + E_2)^* & B_1 - E_1 & -(B_2 - E_2)^* \\ B_2 + E_2 & (B_1 + E_1)^* & B_2 - E_2 & (B_1 - E_1)^* \\ B_1 - E_1 & -(B_2 - E_2)^* & B_1 + E_1 & -(B_2 + E_2)^* \\ B_2 - E_2 & (B_1 - E_1)^* & B_2 + E_2 & (B_1 + E_1)^* \end{pmatrix} = 0 \tag{180}$$

Substitution of the forms in (179) results in this 4×4 matrix equation (180). In ordinary linear algebra, with its associative $\cdot$ product, such an equation as (180) would naturally break out into four separate equations built around each 4×1 column vector appearing in the electromagnetic square matrix. But, in the non-associative algebra, such column vectors are paired together in transformations, so that two column vectors are required to describe the effects of the derivative operator transformation matrix on the electromagnetic fields. This means that the type of questions that makes sense to ask in the usual linear algebra, must be modified to account for this pairing of columns. To enquire about the eigenvalue and eigenvectors for a non-associative transformation matrix may not yield meaningful results, since this makes assumptions about the existence of isolated vectors in the system.

Paradoxically, the "non-associative" algebras actually "associate" column vectors into collections that must move around and transform together as a group, and it is difficult, if not impossible, to "disassociate" these column vectors into isolated objects of independent study. The apparent characteristic introduced by such non-associative algebras, like octonions, into physical theory, is the power to explain that tendency for objects to associate. So, it is the theme of "associativity," but with a twist. Associativity, in the language of algebra, is the exchangeability of sequencing of operations, e.g. $x(yz) = (xy)z$. But, associativity, in the physical world, is that tendency of matter to clump together in various formations. These are not entirely separate and distinct applications of the concept of "associativity," however, since one can be seen to be a natural consequence of the other.

The Maxwell's Equations in (180) can now be re-written using ordinary associative matrix algebra over complex numbers, in terms of the corresponding system of percolated and satellite matrices, as presented before in $(165-169)$. The first two columns of the 4×4 electromagnetic field-matrix are thus linked by the following simultaneous equations.

$$\begin{pmatrix} \frac{d}{dz_1} & -\frac{d}{dz_2^*} & \frac{d}{dz_1} & 0 \\ \frac{d}{dz_2} & \frac{d}{dz_1^*} & 0 & \boldsymbol{\frac{d}{dz_1}} \\ -\frac{d}{dz_1} & 0 & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ 0 & -\boldsymbol{\frac{d}{dz_1^*}} & -\frac{d}{dz_1} & \frac{d}{dz_1^*} \end{pmatrix} \begin{pmatrix} B_1+E_1 \\ B_2+E_2 \\ B_1-E_1 \\ B_2-E_2 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \boldsymbol{\frac{d}{dz_2}} & 0 \\ 0 & 0 & 0 & \boldsymbol{\frac{d}{dz_2}} \\ -\boldsymbol{\frac{d}{dz_2}} & 0 & 0 & 0 \\ 0 & -\boldsymbol{\frac{d}{dz_2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} -(B_2+E_2)^* \\ (B_1+E_1)^* \\ -(B_2-E_2)^* \\ (B_1-E_1)^* \end{pmatrix} = 0 \tag{181}$$

$$\begin{pmatrix} 0 & 0 & -\boldsymbol{\frac{d}{dz_2^*}} & 0 \\ 0 & 0 & 0 & -\boldsymbol{\frac{d}{dz_2^*}} \\ \boldsymbol{\frac{d}{dz_2^*}} & 0 & 0 & 0 \\ 0 & \boldsymbol{\frac{d}{dz_2^*}} & 0 & 0 \end{pmatrix} \begin{pmatrix} B_1+E_1 \\ B_2+E_2 \\ B_1-E_1 \\ B_2-E_2 \end{pmatrix} + \begin{pmatrix} \frac{d}{dz_1} & -\frac{d}{dz_2^*} & \boldsymbol{\frac{d}{dz_1^*}} & 0 \\ \frac{d}{dz_2} & \frac{d}{dz_1^*} & 0 & \frac{d}{dz_1^*} \\ -\boldsymbol{\frac{d}{dz_1^*}} & 0 & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ 0 & -\frac{d}{dz_1^*} & \frac{d}{dz_2} & \frac{d}{dz_1^*} \end{pmatrix} \begin{pmatrix} -(B_2+E_2)^* \\ (B_1+E_1)^* \\ -(B_2-E_2)^* \\ (B_1-E_1)^* \end{pmatrix} = 0 \tag{182}$$

These two equations, (181) and (182), form a linear system of 4×4 matrix equations that contain the full content of the Maxwell Equations. The second pair of equations below, (183) and (184), simply duplicate this information, re-writing the equations in an alternative manner, since we constructed the initial 2×2 quatro-quaternionic electromagnetic field-matrix in (177) by doubling up the original electromagnetic equations. The doubling up requires the presence of these additional "shadow parameters," that duplicate the information content, to track around with the original field parameters, in order to facilitate matrix calculations within a square matrix format.

$$\begin{pmatrix} \frac{d}{dz_1} & 0 & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ 0 & \boldsymbol{\frac{d}{dz_1}} & \frac{d}{dz_2} & \frac{d}{dz_1^*} \\ -\frac{d}{dz_1} & \frac{d}{dz_2^*} & \frac{d}{dz_1} & 0 \\ -\frac{d}{dz_2} & -\frac{d}{dz_1^*} & 0 & \boldsymbol{\frac{d}{dz_1}} \end{pmatrix} \begin{pmatrix} B_1-E_1 \\ B_2-E_2 \\ B_1+E_1 \\ B_2+E_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\frac{d}{dz_2}} & 0 & 0 & 0 \\ 0 & \boldsymbol{\frac{d}{dz_2}} & 0 & 0 \\ 0 & 0 & \boldsymbol{\frac{d}{dz_2}} & 0 \\ 0 & 0 & 0 & \boldsymbol{\frac{d}{dz_2}} \end{pmatrix} \begin{pmatrix} -(B_2-E_2)^* \\ (B_1-E_1)^* \\ -(B_2+E_2)^* \\ (B_1+E_1)^* \end{pmatrix} = 0 \tag{183}$$

$$\begin{pmatrix} -\boldsymbol{\frac{d}{dz_2^*}} & 0 & 0 & 0 \\ 0 & -\boldsymbol{\frac{d}{dz_2^*}} & 0 & 0 \\ 0 & 0 & -\boldsymbol{\frac{d}{dz_2^*}} & 0 \\ 0 & 0 & 0 & -\boldsymbol{\frac{d}{dz_2^*}} \end{pmatrix} \begin{pmatrix} B_1-E_1 \\ B_2-E_2 \\ B_1+E_1 \\ B_2+E_2 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\frac{d}{dz_1^*}} & 0 & \frac{d}{dz_1} & -\frac{d}{dz_2^*} \\ 0 & \frac{d}{dz_1^*} & \frac{d}{dz_2} & \frac{d}{dz_1^*} \\ -\frac{d}{dz_1} & \frac{d}{dz_2^*} & \boldsymbol{\frac{d}{dz_1^*}} & 0 \\ -\frac{d}{dz_2} & -\frac{d}{dz_1^*} & 0 & \frac{d}{dz_1^*} \end{pmatrix} \begin{pmatrix} -(B_2-E_2)^* \\ (B_1-E_1)^* \\ -(B_2+E_2)^* \\ (B_1+E_1)^* \end{pmatrix} = 0 \tag{184}$$

But, the general idea of importance here, is that every non-associative matrix equation in $M_{[\times]}(4,\mathbb{C})$ can be re-written as a system of associative matrix equations, and thus solved by the usual methods of the standard matrix algebra. The same is true for the non-associative 8×8 algebra over reals, $M_{[\times]}(8,\mathbb{R})$, only there the number of simultaneous equations in a set doubles to 4, owing to the 4 column vectors that link up and transform as a group. The non-associative algebras can then be viewed as a condensed representation of such systems of associative simultaneous linear equations. The Maxwell's Equations example illustrates how a field-matrix might be employed in this context.

`Conjecture:` $\mathbb{O} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$. i.e. the product algebra of the octonion algebra with itself is isomorphic to the non-associative 8×8 matrix algebra over the reals defined by the *derived twisted product* $\times$ given in APPENDIX A.
`Explore:`
Let an octonion be, $o = o_u e_u$, with basis, $e_u,\ u = 0, 1, 2, ..., 7$. An inspection of the octonion o in the 8×8 matrix representation (117) shows that it can also be written,

$$o = ae_0 + be_1 + ce_2 + de_3 + ee_4 + fe_5 + ge_6 + he_7. \tag{185}$$

$$e_0 = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{pmatrix} \quad e_1 = \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix} \quad e_2 = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix} \quad e_3 = \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$e_4 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_5 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_6 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad e_7 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{186}$$

This gives us a convenient basis to start with. While an octonion basis can be established in several alternative ways, our matrix in (117) is obtained simply by expanding through repeated substitution of the 2×2 representation form in (5) from the Cayley-Dickson (I) construction. To show that e_u are a basis, we compute the $e_u \times e_v$ products, using the derived twisted product for these 8×8 matrices. This particular $M_{[\times]}(8, \mathbb{R})$ octonion basis is then found to have the identical product table to that previously given in (153) for the octonion $M_{[\times]}(4, \mathbb{C})$ representation, when the basis matrices have the e_u label assignments shown above. As far as "octonion representation" is concerned, it does not matter which definition for the $\times$ product we choose from APPENDIX A. Both definitions give identical octonion tables.

We can expand this basis set in a number of ways. One method is to modify the signs on these $e_u,\ u = 1, 2, ..., 7$ matrices to get other indepdenent matrices whose square is still -1, and then take the products of the new with the old to find the new basis elements, and repeat the procedure until we get a complete set of 64 basis matrices. Another method is to take the 32 complex basis matrices found for $M_{[\times]}(4, \mathbb{C})$ in (148, 155) and expand these 4×4 complex to 8×8 real by substituting 2×2 real matrices for the internal 0's, 1's, and i's, then identify one more independent 8×8 matrix to multiply these 32 by to double the set to 64. This indicates, also, that half the elements of $M_{[\times]}(8, \mathbb{R})$ should form a subalgebra isomorphic to $\ddot{\mathbb{H}} \times \mathbb{O}$, notwithstanding the fact that the $\times$ product definition is somewhat different between the $M_{[\times]}(4, \mathbb{C})$ algebra and the $M_{[\times]}(8, \mathbb{R})$ algebra. Therefore, we can optionally pick an initial alternating complex number triple $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \in \ddot{\mathbb{H}}$, independent of the initial e_u basis of (186), and expand to 32 elements by taking products, then find one more independent matrix to get to 64. But, this time we also have several independent quaternion triples, $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\} \in \mathbb{H}$, outside the initial octonion basis, so we can start with $\mathbb{H} \times \mathbb{O}$ constructions instead of $\ddot{\mathbb{H}} \times \mathbb{O}$. We couldn't represent $\mathbb{H} \times \mathbb{O}$ using the $M_{[\times]}(4, \mathbb{C})$ before, so given that this is new and interesting we shall choose this approach to expand our basis set.

With a bit of exploration, we find and select the following independent $\boldsymbol{ijk}$ quaternion triple, using the "second" definition for the $\times$ product given in APPENDIX A, labeled (ALL CAYLEY-DICKSON ALGEBRAS):

$$\mathbf{1} = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{pmatrix} \quad \boldsymbol{i} = \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \end{pmatrix} \quad \boldsymbol{j} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \quad \boldsymbol{k} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$q = w + x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \qquad w, x, y, z, \in \mathbb{R}\ , \quad q \in \mathbb{H} \tag{187}$$

PERCOLATION ISSUES: Once we're no longer just representing the simple octonion algebra, $\mathbb{O}$, we have to decide exactly how to extend the definition of the $\times$ product, because there are differences in the generalized algebras that result from the choice we make. An inspection of the two definitions for the 8-dim $\times$ product, given in APPENDIX A, reveals they have two different "percolation" profiles. If we select the first definition, for example, and use that for the definition of the generalized matrix algebra $M_{[\times]}(8,\mathbb{R})$, then we find there are 16 independent quaternion triples within $M_{[\times]}(8,\mathbb{R})$ outside of the initial octonion basis set. But, if we select the second definition, we find there are only 14 quaternion triples outside the initial octonion basis set. If we think of the $M_{[\times]}(8,\mathbb{R})$ algebra as, in some sense, resulting from the doubling of the $M_{[\times]}(4,\mathbb{C})$ algebra, then the 64 basis elements of the former should, in some sense again, contain 2 copies of the 32 elements found in the latter. But, $M_{[\times]}(4,\mathbb{C}) \cong \ddot{\mathbb{H}} \times \mathbb{O}$, and there are exactly 7 quaternion triples provided by $\mathbb{O}$ and none otherwise. So, we'd anticipate finding another 7 quaternion triples outside the the initial octonion basis in $M_{[\times]}(8,\mathbb{R})$. In which case, finding 16 triples seems a bit odd, and 14 triples, while more than 7, are at least a multiple of 7. This leads us to be inclined to select the same definition for the $\times$ product that is required for all Cayley-Dickson algebras to use here also for the $M_{[\times]}(8,\mathbb{R})$ algebra.

Notice that when we construct the 4×4 derived twisted $\times$ product, it doesn't matter whether we use $M(2,\mathbb{C})$ or $M_{[\times]}(2,\mathbb{C})$, we not only obtain the same 4×4 representation for the octonions, but the generalized algebra, $M_{[\times]}(4,\mathbb{C})$, is also independent of choice of product definition. This is because the "commuting" complex number entires precipitate the same percolation profile in either case, resulting in only one effective definition for the $\times$ product. But, when we make this 2×2 subtitution "twice" in succession, to expand our matrix algebra to 8×8, we're then replacing the "commuting" complex numbers with "non-commuting" 2×2 matrices in the final step, and so it does matter now which method is used to construct the derived twisted product. We actually have 4 choices to go from 2×2 to 8×8. We present the definitions that result from two of those in the appendix; first definition uses $M(2,\mathbb{C})$ followed by $M(2,\mathbb{R})$, while second definition uses $M_{[\times]}(2,\mathbb{C})$ followed by $M_{[\times]}(2,\mathbb{R})$. But, we could also follow a third path, $M_{[\times]}(2,\mathbb{C})$ followed by $M(2,\mathbb{R})$, or fourth method, use $M(2,\mathbb{C})$ followed by $M_{[\times]}(2,\mathbb{R})$. From the point of view of representing octonions, all these definitions produce the same 8×8 matrix representation. But, when considering the generalized matrix algebra that contains the octonions, these definitions have a different impact on the structure of that matrix algebra. We won't discuss all the nuances between these alternatives here. Instead, we've picked one to illustrate this study.

THE $\mathbb{H} \times \mathbb{O}$ SUBALGEBRA. Using the quaternion defined in (187) and the octonion basis in (186), we now construct the subalgebra $\mathbb{H} \times \mathbb{O}$, by taking the products of the two bases to expand the set of elements. With the quaternion multiplying from the left, we obtain a set of 32 basis matrices, $\beta_u,\ u = 0, 1, ...31$, that is closed under the twisted $\times$ product: $\beta_u = e_u,\ \beta_{u+8} = \boldsymbol{i} \times e_u,\ \beta_{u+16} = \boldsymbol{j} \times e_u,\ \beta_{u+24} = \boldsymbol{k} \times e_u,\ u = 0, 1, 2, 3, 4, 5, 6, 7$ (see APPENDIX B).

$$\boldsymbol{l} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \tag{188}$$

To get the remaining basis matrices for $M_{[\times]}(8,\mathbb{R})$, we identify one more suitable 8×8 matrix that is also independent of these 32. The matrix $\boldsymbol{l}$ shown in (188), has a simple cross diagonal form, with $\boldsymbol{l}^2 = +1$, is independent of the other basis elements, and allows us to obtain the remaining 32 basis matrices to complete the set of 64. We multiply with $\boldsymbol{l}$ from the left, and results are shown on PAGE B of APPENDIX B.

$$\mathbb{H} \times \mathbb{O} \ + \ \boldsymbol{l} \times (\mathbb{H} \times \mathbb{O}) \ \cong \ M_{[\times]}(8,\mathbb{R}) \tag{189}$$

This allows us to write the generalized matrix algebra as a particular "doubling" of the $\mathbb{H} \times \mathbb{O}$ algebra, as shown in (189), or to equivalently construct a new type of " *twisted split-octonion* ", $\tilde{\ddot{\mathbb{O}}} = \mathbb{H} + \boldsymbol{l} \times \mathbb{H}$, from Hamilton's quaternions, and use the products of the basis elements, from this new 8-dim hypercomplex number and the regular octonions, to arrive at the same result. The "split-octonion", $\ddot{\mathbb{O}}$, is a variation of John Grave's octonions, where the doubling from Hamilton's quaternions is done by the introduction of a positive square imaginary element, $l^2 = +1$, instead of the more usual complex imaginary, $i^2 = -1$, customary for octonions. A split-octonion basis can be represented by the eight elements, $\beta_u,\ u = 0, 1, 2, .., 7$, given in sequence by, $\{1,\ i,\ j,\ k,\ l,\ li,\ lj,\ lk\}$, where the rules for multiplication are given by $i, j, k \in \mathbb{H},\ l^2 = +1,\ li = -il,\ lj = -jl,\ lk = -kl$, etc. In compact form the product table is given below on the left in table 190. The "twisted split-octonions", $\tilde{\ddot{\mathbb{O}}}$, can be similarily labeled, $\beta_u,\ u = 0, 1, 2, .., 7$, given in sequence by, $\{\, 1,\ i,\ j,\ k,\ l,\ l \times i,\ l \times j,\ l \times k \,\}$, where the rules for multiplication are given by, $i, j, k \in \mathbb{H},\ l^2 = +1,\ l \times i = -i \times l,\ l \times j = -j \times l,$ but then, $l \times k = +k \times l$, etc. In compact form the product table is given below on the right in table (190).

×	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	$\bar{0}$	3	$\bar{2}$	$\bar{5}$	4	$\bar{7}$	6
2	2	$\bar{3}$	$\bar{0}$	1	$\bar{6}$	7	4	$\bar{5}$
3	3	2	$\bar{1}$	$\bar{0}$	$\bar{7}$	$\bar{6}$	5	4
4	4	5	6	7	0	1	2	3
5	5	$\bar{4}$	$\bar{7}$	6	$\bar{1}$	0	3	$\bar{2}$
6	6	7	$\bar{4}$	$\bar{5}$	$\bar{2}$	$\bar{3}$	0	1
7	7	$\bar{6}$	5	$\bar{4}$	$\bar{3}$	2	$\bar{1}$	0

$\ddot{\mathbb{O}}$ = SPLIT-$\mathbb{O}$

$(+ + + + - - - -)$

$$(\mathbb{H} \ + \ l \times \mathbb{H}) \times \mathbb{O} \ \cong \ M_{[\times]}(8, \mathbb{R})$$

$$\tilde{\ddot{\mathbb{O}}} \ = \ (\mathbb{H} \ + \ l \times \mathbb{H}) \ \text{"}\textit{Twisted Split-Octonions}\text{"} \longrightarrow$$

$$\longleftarrow \ \ddot{\mathbb{O}} \ = \ (\mathbb{H} \ + \ l\,\mathbb{H}) \quad \text{"}\textit{Split-Octonions}\text{"}$$

$$\tilde{\ddot{\mathbb{O}}} \times \mathbb{O} \ \cong \ M_{[\times]}(8, \mathbb{R})$$

×	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	$\bar{0}$	3	$\bar{2}$	5	$\bar{4}$	$\bar{7}$	6
2	2	$\bar{3}$	$\bar{0}$	1	6	7	$\bar{4}$	$\bar{5}$
3	3	2	$\bar{1}$	$\bar{0}$	$\bar{7}$	6	$\bar{5}$	4
4	4	$\bar{5}$	$\bar{6}$	$\bar{7}$	0	$\bar{1}$	$\bar{2}$	$\bar{3}$
5	5	4	7	$\bar{6}$	1	0	$\bar{3}$	2
6	6	$\bar{7}$	4	5	2	3	0	$\bar{1}$
7	7	6	$\bar{5}$	4	$\bar{3}$	2	$\bar{1}$	$\bar{0}$

$\tilde{\ddot{\mathbb{O}}}$ = TWISTED-SPLIT-$\mathbb{O}$

$(+ + + + - - - +)$

(190)

The two split algebras share some common attributes, while there are also some significant differences. They both split the signature of the norm, to contain some + and some − signed elements, but do so differently. Comparing norm signatures, $N(\ddot{\mathbb{O}}) \sim (+ + + + - - - -)$ verses $N(\tilde{\ddot{\mathbb{O}}}) \sim (+ + + + - - - +)$, we see the "twisted" version of the split octonion unevenly divides the number of positive and negative signs in contrast to the even distribution found in the ordinary split octonion algebra. A search of the 14 quaternion triples that exist outside the regular octonion basis in (186) turns up no set of seven triples with a common 7-element basis, so that we cannot construct a second regular octonion in the remaining elements, only this twisted split octonion.

So, given that our initial guess, that $\mathbb{O} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$, turned out to be false, but that our exploration shows we can use a variation of the octonion algebra instead, we now replace our conjecture with the following alternative.

`Conjecture:` $\tilde{\ddot{\mathbb{O}}} \times \mathbb{O} \cong M_{[\times]}(8, \mathbb{R})$. i.e. the product algebra of the twisted split-octonion algebra with the octonion algebra is isomorphic to the non-associative 8×8 matrix algebra over the reals defined by the second definition of the *derived twisted product* $\times$ given in APPENDIX A.

`Proof:` With the octonion basis in (186), the independent quaternion $\boldsymbol{ijk}$ triple in (187), and the additional independent basis matrix, $\boldsymbol{l}$, of (188), we first construct the twisted split-octonion, $\tilde{\ddot{\mathbb{O}}} = \mathbb{H} + \boldsymbol{l} \times \mathbb{H}$, and then take the product of this basis with the octonion basis, $\mathbb{O}$, to obtain the set of 64 matrices. These are easily shown to be linearly independent, and therefore represent the $M_{[\times]}(8, \mathbb{R})$ algebra, since they share the same $\times$ product. The basis matrices so constructed are the same as given in the APPENDIX B, up to a sign, since the $\boldsymbol{l}$ element only associates in half of the cases, while it anti-associates in the other half, in the 24 instances of the product, $\boldsymbol{l} \times (\mathbb{H} \times \mathbb{O}) = \pm(\boldsymbol{l} \times \mathbb{H}) \times \mathbb{O}$, where $\mathbb{H}$ is replaced by an imaginary element of the $\boldsymbol{ijk}$ triple (i.e. excl. 1), and $\mathbb{O}$ is replaced by one of the eight basis elements, $e_u,\ u = 0, 1, 2, .., 7$.

THE TWISTED $\times$ PRODUCT INVERSE FOR $M_{[\times]}(8, \mathbb{R})$.

To construct the inverse of a non-associative matrix in this 8-dim matrix algebra, we proceed in a similar manner to the 4×4 algebra. This time, instead of a pair of coupled equations, the percolation results in sets of four simultaneous matrix equations expressed in ordinary matrix algebra. Each percolated matrix now has three satellite matrices associated with it, making a total of 8 percolated, $A_u,\ u = 1, 2, ..., 8$, and 24 satellites, $S_{u1}, S_{u2}, S_{u3},\ u = 1, 2, ..., 8$, to complete the system of equations. Since these 8×8 matrices are now members of the ordinary real matrix algebra, the usual matrix methods for finding the inverse of a matrix can then be applied to solve the system of equations, similar to that illustrated above in the $M_{[\times]}(4, \mathbb{C})$ situation, and so find the non-associative 8×8 matrix inverse.

IV. CONCLUSIONS.

MOTIVATIONS. This research was initially inspired by the need to find a way to split the product operator for the 16-dim system of `hexpe numbers`. In our previous paper on "*Hexpentaquaternions*" [PJ2][2], we discussed the fact that, since quaternions form a non-abelian algebra, manipulation of even simple algebraic expressions is difficult without a way to effectively commute the variables in a product. We showed how solving linear quaternion equations with matrix algebra reveals what kind of modifications are required to enable the commuting of these factors. The quaternion parameters in a product, $A \cdot B$, could be permuted by changing the hand of one of the factors, $A \cdot B = B' \cdot \hat{A}$, so that if A and B are right-hand quaternions, B' is now a left-hand quaternion. This led us to construct the two-hand quaternion algebra that would allow us to work with both right hand and left hand quaternions in the same system. The hand changing operator $'$ is a very convenient device, and works somewhat similar to the conjugate *, so that the algebra has a rather pleasing feel in the working out of solutions. However, there's one problem with the whole idea. Not only must we change the hand of the B parameter, $B \mapsto B'$, we must also mark the A parameters with a hat, $A \mapsto \hat{A}$, to indicate which variable is moving and which variable is the fixed pivot. The alternative, $A \cdot B = \hat{B} \cdot A'$, is obviously another equally valid way to resolve the commutation. This means we must have two forms for the representation of our quaternion variables, that distinguish between the hat, $\hat{A}$, and hat-free, A, state of a parameter. Matrix algebra resolves this issue by representing the hat, $\hat{A}$, with column vectors, and the hat-free, A, by square matrices. But, this technique is limited in that it can only be used to solve linear problems, since the hat, or column vectors, are generally required to stand on one side of the expressions. So, it becomes difficult to extend the method to the manipulation of expressions consisting of polynomials with degree higher than one.

`Splitting the product.` We raised the point, in our previous paper, that there are two ways to resolve this commutation issue; one could either "split the representation" into two, or "split the product operator" into two. Matrix algebra shows immediately how to split the representation, but it is not equally obvious how to split the product operator.

$$A \cdot B = AB \tag{191}$$

$$A' \odot B = BA \tag{192}$$

Splitting the product would allow us to keep the same representation for all quaternion parameters. We'd still have our hand changing operator $'$, and so two-hand quaternions, but we wouldn't be constrained by the column vector effect, to have some of our parameters stuck on the right-most side of our algebraic expressions. We therefore need two forms for the product operator, $\cdot$ and $\odot$, with the kind of rules shown above (In discussing this concept, we used the symbol $\otimes$ in our previous paper to represent that unknown alternative product, but we use $\odot$ here instead to avoid confusion with the *tensor product* of algebras).

Having such a dual product algebra would allow us to commute the variables, and write, $A \cdot B = B' \odot A$, with a simple hand change and operator change. But, we'd need to establish the working rules for the two multiplication operations being used in the same system. Our single product, AB, of the usual algebra, is split into two forms, $A \cdot B$ and $A \odot B$, to construct the method. The first form, $A \cdot B$, obeys the all the usual rules for the multiplication operator, but the second, $A \odot B$ and $A' \odot B$, must have some unusual rules that allow it to work. Attempts to find the right set of rules for this second operator have proved illusive so far. We first tried a number of heuristic approaches and elementary conjectures, but they all turned out to have seemingly unresolvable problems. Our initial naive approach was to simply take the four—commutive, associative, right distributive, and left distributive—laws for pivots, that work so well, and use these to guess the corresponding dual product rules that must apply to achieve the same effects.

$$qB = B'\hat{q} \qquad\qquad q \cdot B = B' \odot q \tag{193}$$

$$A(B'\hat{q}) = (AB')\hat{q} \qquad\qquad A \cdot (B' \odot q) = (A \cdot B') \odot q \tag{194}$$

$$G\hat{q} + F\hat{q} = (G+F)\hat{q} \qquad\qquad G \odot q + F \odot q = (G+F) \odot q \tag{195}$$

$$H(G\hat{q} + F\hat{p}) = (HG)\hat{q} + (HF)\hat{p} \qquad\qquad H \cdot (G \odot q + F \odot\ p) = (H \cdot G) \odot q + (H \cdot F) \odot p \tag{196}$$

$$H, G, F, \in \mathbb{X}_n\ , \quad \texttt{with either} \quad q, p, A, B, \in \mathbb{H}_R\ , \quad \texttt{or} \quad q, p, A, B \in \mathbb{H}_L$$

So, for each given pivot law (above left) we introduced the corresponding dual product, $\cdot$ and $\odot$, form (above right). We then spent a considerable effort trying to "fix" these equivalent dual product rules, to no avail. There seemed to be lots of problems establishing a dual product algebra, and getting the two multiplications to work together appeared too difficult to resolve by just guesswork. Then the thought occured to us that the octonions have a non-associative product, and matrices have an associative product. So, if one could figure out how to modify the product in matrix algebra to represent octonions, one would have a natural dual product algebra to work with. This would provide a more structured approach to searching for that dual product system we required. It would also avoid the arbitrary guesswork that marked some of our other approaches. And certainly, a study of a dual product system should give us clues how to proceed. A search of the literature, using our limited resources, did not turn up anything we could use. But the method of constructing the *twisted product* for matrix algebra did occur to us along the way, and was a partial success.

We could now construct a matrix algebra with two products, $\cdot$ and $\times$, one associative and the other non-associative. Well, our two-hand quaternions are already represented by a matrix algebra, so could the new matrix product $\times$ from the octonion solution be adapted to play the role of the $\odot$ we're looking for? Starting with our derived complex matrix algebra, $M_{[\times]}(4, \mathbb{C})$, and restricting the complex numbers to those with vanishing imaginary parts, we obtain a corresponding matrix algebra over reals, $M_{[\times]}(4, \mathbb{R})$. The *twisting* in the definition of the $\times$ has no effect here on our real numbers, but the *percolation* does, and so our $\times$ product is different from the standard $\cdot$ product. Now we can explore the effect this derived twisted product has on our `hexpe number` system, by extending that algebra to include this $\times$ product. The first surprising observation is that, despite the percolation, the set of 32 hexpe basis numbers is closed under this new operator, i.e. $\forall\ h, g \in \mathbb{X}_b\ :\ h \times g \in \mathbb{X}_b$. The second surprise, is that right hand quaternions commute, $p, q \in \mathbb{H}_R\ :\ p \times q = q \times p$. However, the operation is not closed on the set of right hand quaternions, since the product of two right hand quaternions is not usually another right hand quaternion, it's a general hexpe number instead. The $\times$ product table for the hexpe numbers is given in APPENDIX C. This can be compared with the associative product table (TABLE T.2) on pg. 60 of our previous paper [PJ2][2] (The "$\times$" symbol in the upper left corner of the table in that paper should now read "$\cdot$" instead).

Given the rules (193 – 196) above, that describe what sort of system we're aiming for, the $\times$ product seems not to be the right operator. However, we also discovered, in the process of constructing the general quatro-quaternion algebra, that the new non-associative operator itself seems to require a dual form. This operator has two forms, that we could refer to as "forward" and "reverse" products, $\times$ and $\overline{\times}$, which, may yet yield some further insights that we could use to help construct our system.

STITCHING PATTERNS. The standard matrix algebra was constructed from systems of linear equations having abelian parameters in mind. So, what does it mean to use non-abelian parameters for matrix components? In the 1800s, Sir W. R. Hamilton engaged in a running battle with mathematicians over the meaning, interpretation, and development of algebra. Hamilton's view was that algebra should always be based on reality, so that algebraic constructs could have physical application. Sir Hamilton wrote a celebrated essay on "Algebra as the Science of Pure Time" expounding his point of view, and valiantly defended his arguments in various other writings. But, he eventually lost this battle to the followers of Boole, who viewed logic as the very foundation of algebra, and were happy to construct any kind of abstraction, however useless it might be in physical interpretation, provided the construction was internally consistent with regard to the fundamental axioms and laws of logic. Hamilton may have lost this battle, but his musings often strike a cord with Physicists attempting to find the right way to apply the mathematician's construction to their physical problems. What does it mean to write $M(2, \mathbb{H})$, for example?

If we consider a system of linear equations,

$$\begin{aligned}
A_{00}x_0 + A_{01}x_1 + A_{02}x_2 + A_{03}x_3 &= y_0 & \qquad B_{00}y_0 + B_{01}y_1 + B_{02}y_2 + B_{03}y_3 &= z_0 \\
A_{10}x_0 + A_{11}x_1 + A_{12}x_2 + A_{13}x_3 &= y_1 & \qquad B_{10}y_0 + B_{11}y_1 + B_{12}y_2 + B_{13}y_3 &= z_1 \\
A_{20}x_0 + A_{21}x_1 + A_{22}x_2 + A_{23}x_3 &= y_2 & \qquad B_{20}y_0 + B_{21}y_1 + B_{22}y_2 + B_{23}y_3 &= z_2 \\
A_{30}x_0 + A_{31}x_1 + A_{32}x_2 + A_{33}x_3 &= y_3 & \qquad B_{30}y_0 + B_{31}y_1 + B_{32}y_2 + B_{33}y_3 &= z_3
\end{aligned} \tag{197}$$

where we have to solve many such sets over and over again, the advantages of matrix algebra are obvious. Matrices allow us to *separate* the known parameters from the unknowns, *aggregate* the like parameters into collections, and *abbreviate* these expressions, achieving a wondrous efficiency in economy of symbols. At the same time, this whole construction process simultaneously generates a new perception of the algebra of these linear expressions, by revealing the hidden algebraic structure that becomes really clear only through this *abstraction* provided by the matrix formulation. We can separate, aggregate, and abbreviate these expressions thus,

$$a = \begin{pmatrix} A_{00} & A_{01} & A_{02} & A_{03} \\ A_{10} & A_{11} & A_{12} & A_{13} \\ A_{20} & A_{21} & A_{22} & A_{23} \\ A_{30} & A_{31} & A_{32} & A_{33} \end{pmatrix} \quad b = \begin{pmatrix} B_{00} & B_{01} & B_{02} & B_{03} \\ B_{10} & B_{11} & B_{12} & B_{13} \\ B_{20} & B_{21} & B_{22} & B_{23} \\ B_{30} & B_{31} & B_{32} & B_{33} \end{pmatrix} \quad x = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \end{pmatrix} \quad z = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} \tag{198}$$

$$a \cdot x = y \ , \quad b \cdot y = z \tag{199}$$

This allows us to use pre-established formulas for the inverse of a matrix to write, $x = a^{-1} \cdot y$ and $y = b^{-1} \cdot z$, for example, or to write the z's in terms of the x's, $z = b \cdot (a \cdot x) = (b \cdot a) \cdot x$, by suitable extension of the definition of the matrix product $\cdot$ to square arrays, which lets us easily recognise the algebraic structure property, $b \cdot a \neq a \cdot b$, even though the components of the matrices themselves commute, and so on. But, this is all constructed on the premise that we can *separate* the parameters, A's from x's etc., pack them into these arrays, and then " *stitch* " them back together again to recover the linear expressions using the standard definition of the matrix product. Our ability to unzip the original linear expressions, and zip them back together correctly, is largely dependent on the fact that these component parameters are abelian factors. Now consider another somewhat similar system of linear equations,

$$\begin{aligned}
A_{00}x_0 + x_1A_{01} + x_2A_{02} + A_{03}x_3 &= y_0 & \qquad B_{00}y_0 + y_1B_{01} + y_2B_{02} + B_{03}y_3 &= z_0 \\
A_{10}x_0 + x_1A_{11} + x_2A_{12} + A_{13}x_3 &= y_1 & \qquad B_{10}y_0 + y_1B_{11} + y_2B_{12} + B_{13}y_3 &= z_1 \\
A_{20}x_0 + x_1A_{21} + x_2A_{22} + A_{23}x_3 &= y_2 & \qquad B_{20}y_0 + y_1B_{21} + y_2B_{22} + B_{23}y_3 &= z_2 \\
A_{30}x_0 + x_1A_{31} + x_2A_{32} + A_{33}x_3 &= y_3 & \qquad B_{30}y_0 + y_1B_{31} + y_2B_{32} + B_{33}y_3 &= z_3
\end{aligned} \tag{200}$$

If the parameters are still abelian, we can always rearrange these equations (200) to match the form (197), and apply our standard matrix algebra. But, what happens when the parameters are non-abelian? The linear expressions in (197) and (200) have the following twisted product forms,

R	+	R	+	R	+	R	R	+	L	+	L	+	R
R	+	R	+	R	+	R	R	+	L	+	L	+	R
R	+	R	+	R	+	R	R	+	L	+	L	+	R
R	+	R	+	R	+	R	R	+	L	+	L	+	R

The expressions in (197) are based on a system of right actions, R+R+R+R, while those in (200) have a combination of right and left action forms, R+L+L+R. If we separate and collect the A's and x's into arrays, how do we zip them back together again?

The usual matrix product does not have the correct stitching pattern recorded in its method, that would enable it to reproduce the twisted profile of a general non-abelian linear expression. So, before we use non-abelian parameters for the components of a matrix we should expect to have to update our very conception of matrix algebra.

Certainly, in the limited case, when our non-abelian parameters happen to form right action expressions that match the form in (197), we can apply the standard matrix algebra to this special situation, and so extend the definition to $M(2,\mathbb{H})$. But, those very non-abelian parameters can form many alternative expressions also, like the Cayley-Dickson construction process exhibits, that are more natural forms in their own algebra, and there's no reason to restrict the consideration to just one type of linear expression form. The quatro-quaternions provide us with a simple and natural way to extend the matrix algebra, to account for *some* other forms found in non-abelian mathematics.

SUMMARY. We found and reviewed two papers on the topic of "Matrix Representations of Octonions" in the hope that we could use some existing art for our work. Daboul and Delbourgo [DD][3] introduce an extension to the Zorn's vector matrix algebra, while Tian [YT][4] presents a development based on the doubling of 4×4 real matrix representations of Hamilton's quaternions. We ended up introducing our own method, which seems a more natural construction to us, because it tackles the problem at the "foundations" of the very concept of the definition of matrix multiplication itself, rather than attempting to adapt the existing structure by grafting a special product on at a higher level. Octonions can be represented by 2×2, 4×4, and 8×8 matrices, as we have demonstrated here, and so can any Cayley-Dickson algebra. By repeated substitution of the 2×2 matrix form, using the non-associative matrix product, any Cayley-Dickson algebra can be ultimately represented as a matrix algebra over reals. In each substitution, however, the matrix algebra constructed forms a more general algebra than the Cayley-Dickson algebra it contains, and with the exception of the quatro-quaternions, $\mathbb{QQ} = M_{[\cdot,\times]}(2,\mathbb{H})$, and a brief review of some characteristics of the descendant $M_{[\times]}(4,\mathbb{C})$ and $M_{[\times]}(8,\mathbb{R})$ matrix algebras derived from them, the properties of these more general matrix algebras have not been examined in this paper. We chose to focus on the quatro-quaternions, since this is the simplest non-trivial example of the concept of the twisted product that might be of general interest.

`Product Algebras:` We showed that $\tilde{\ddot{\mathbb{O}}} \times \mathbb{O} \cong M_{[\times]}(8,\mathbb{R})$, $\ddot{\mathbb{H}} \times \mathbb{O} \cong M_{[\times]}(4,\mathbb{C})$, and, $\mathbb{C} \times \mathbb{O} \cong M_{[\times]}(2,\mathbb{H})$, but admit there are probably more rigorous ways to prove these identities than we have given in this paper. The idea to search for these identities was stimulated by the known results, $\mathbb{H}' \otimes \mathbb{H} \cong M(4,\mathbb{R})$ and $\mathbb{H} \otimes \mathbb{H} \cong M(4,\mathbb{R})$, which was pointed out to us recently on `sci.math.research`, and while the "tensor product" itself could not be used in this context, because of the very "twisted" nature of the non-associative algebras, we found we could nevertheless construct these kinds of alternative "product algebras" whenever the two hypercomplex algebras involved are able to be represented by a common non-associative matrix algebra.

`Notation:` In the event that other 2×2 twisted product definitions prove useful, we define, $\times_j$, $j = 1, 2, ..., 256$, the indexed operator, sequenced in any particular order, to facilitate representation of the idea, $a \times_j b$, all of which lead to problems solvable with the two-hand quaternion algebra. In this case, $a \overleftarrow{\times}_j b$, refers to the mirror image product that reverses each pair of factors appearing within the matrix product definition expression, and our algebra is now the *generalized quatro-quaternions*, $\mathbb{QQ}_G$. One then thinks of an algebra as possessing a whole category of products of a particular type, rather than a single product, since the single product idea benefited largely from the degenerate nature of the permutation symmetry found in product expressions operating in an environment of abelian factors.

`Matrix Upgrade:` When complex numbers were introduced to matrices, the whole of matrix algebra recieved a minor overhaul, and matrices became Hermitian and Unitary. Complex numbers brought with them that property of *conjugation*, and the new characteristic was quickly integrated into the structure of matrix algebra. One could now take the conjugate * of a matrix as a whole, and combine * with the usual matrix transpose ^{T} to construct the even more useful "hermitian conjugate" †, with $A^\dagger = A^{*T}$, to obtain useful results like $(AB)^\dagger = B^\dagger A^\dagger$, etc.

The result was a complex matrix algebra that proved extraordinarily suitable in applications to physics. But, since quaternions were introducted, even though they also bring something additionally new with them, no similar further upgrade to the matrix algebra has been yet undertaken. The quaternions were forced to work along with the same principles that govern the lower dimensional algebras, when wearing the matrix garments, and their unique character and requirements went unmet. It was as if complex numbers were just tossed into a matrix where conjugation was not allowed, since real numbers did not require them, and so were forced to work with the principles of real algebra in that context. No efforts were made to accomodate the unique properties that the quaternions bring with them. The Cayley-Dickson construction suggests to us that to work with these higher algebras it may be profitable to at least consider both *twisting* and *conjugation*: the twisting facilitates the expression of the non-commuting property starting with the quaternions (i.e. to accomodate the wider variety of alternative non-abelian expressions generally encountered when reckoning with non-commuting parameters), just as the conjugation assists with norms and inversions beginning with the complexes. Non-commuting factors break the permutation symmetry of products existing in mathematical expressions and introduce a diverse multiplicity that needs to be recognised and adequately addressed if one is to seriously consider using such non-abelian parameters within matrices. Even when just working with only matrices over reals, the common practice of placing matrices within matrices raises this same issue of replacing abelian entries with non-abelian entries, and the automatic implicit "right action" convention applied to product expressions in such cases artificially limits the expressiveness of the technique. The new formal twisted $\times$ product definition given in (2) can thus be viewed as a first step on this path towards upgrading the matrix algebra to account for the peculiar characteristics encountered in number systems beyond the reals and complexes.

`Concepts:` We introduced the concepts of *twisting* and *percolation* to throw light on the type of modfications to matrix products, that might, in general, lead to more constructive forms and alternate matrix algebras, and mentioned our quest for the right formulation of that "Algebra of the Split Operator" that led us towards this research. While we have not solved this problem of finding the right operator, we are yet to try the exhaustive search through the twisted and percolated forms to find out what is possible, and cannot claim to have examined and comprehended the simple quatro-quaternions (and their descendent matrix algebras) in sufficient depth to recognise whether or not the answer might somehow be found right here. We introduced the reversing operator, $X = PQ$, $\overleftarrow{X} = QP$, which can be used in various contexts to facilitate discussion and representation of a recurring characteristic in non-abelian expressions that often require simple reversal of two factors to simplify terms or describe modifications in the working out of problems. We introduced the operators, $R(\,\cdot\,)$ and $L(\,\cdot\,)$, that extract the "right pure quaternion" and "left pure quaternion" of a number in two-hand quaternions, similar to the vector operator, $V(\,\cdot\,)$, from Hamilton's calculus; and also extended the concept of conjugation, h^*, to include the right conjugate, h^{*_R}, and left conjugate, h^{*_L}, of a number, that attack the right-hand and left-hand parts separately, to simplify the art of reckoning with hexpe numbers. The usual number systems—*complex*, *quaternion*, and *octonion*—can not only be " *split* " but now " *twisted* ", and either or both, producing alternative varieties with differing norm signatures and product tables, and corresponding *bi*-numbers can be constructed. We introduced "*twisted bi-octonions*," and "*twisted split-octonions*," in the context of the identification of the generalized non-associative matrix algebras, that serve to illustrate two of these additional number varieties.

Acknowledgments

We wish to thank William C. Waterhouse, Creighton Dement, and Prof. Edwin Clark, whose tensor product comments on `sci.math.research`, appearing either directly or indirectly, on our last hexpe paper, led us to concieve of and conjecture the links in connection with the products of algebras to these non-associative matrix algebras.

A.1

APPENDIX A

THE DERIVED TWISTED $\times$ PRODUCT FOR THE 8×8 MATRIX.

The definition of the twisted $\times$ product for $M_{[\times]}(8,\mathbb{R})$. The version given below is obtained from $M_{[\times]}(2,\mathbb{H})$, by replacing the quaternion entries with $M_{[\cdot]}(2,\mathbb{C})$, then using $M_{[\cdot]}(2,\mathbb{R})$ to replace those complex numbers. The definition is valid for representation of octonions only. The alternative definition, immediately following this one, may be used for both octonions and general Cayley-Dickson Algebras. Percolated terms are marked in **boldface**.

$$[C_{ij}] = [A_{pq}] \times [B_{rs}]$$

THE 8-DIM $\times$ PRODUCT (FOR OCTONIONS ONLY):

$$
\begin{aligned}
C_{00} &= A_{00}B_{00} + A_{01}B_{10} + A_{02}B_{20} + A_{03}B_{30} + B_{40}A_{04} + \boldsymbol{B_{41}A_{14}} + \boldsymbol{B_{42}A_{24}} + \boldsymbol{B_{43}A_{34}} \\
C_{10} &= A_{10}B_{00} + A_{11}B_{10} + A_{12}B_{20} + A_{13}B_{30} + \boldsymbol{B_{50}A_{04}} + \boldsymbol{B_{51}A_{14}} + \boldsymbol{B_{52}A_{24}} + \boldsymbol{B_{53}A_{34}} \\
C_{20} &= A_{20}B_{00} + A_{21}B_{10} + A_{22}B_{20} + A_{23}B_{30} + \boldsymbol{B_{60}A_{04}} + \boldsymbol{B_{61}A_{14}} + \boldsymbol{B_{62}A_{24}} + \boldsymbol{B_{63}A_{34}} \\
C_{30} &= A_{30}B_{00} + A_{31}B_{10} + A_{32}B_{20} + A_{33}B_{30} + \boldsymbol{B_{70}A_{04}} + \boldsymbol{B_{71}A_{14}} + \boldsymbol{B_{72}A_{24}} + \boldsymbol{B_{73}A_{34}} \\
C_{40} &= B_{00}A_{40} + \boldsymbol{B_{01}A_{50}} + \boldsymbol{B_{02}A_{60}} + \boldsymbol{B_{03}A_{70}} + A_{44}B_{40} + A_{45}B_{50} + A_{46}B_{60} + A_{47}B_{70} \\
C_{50} &= \boldsymbol{B_{10}A_{40}} + \boldsymbol{B_{11}A_{50}} + \boldsymbol{B_{12}A_{60}} + \boldsymbol{B_{13}A_{70}} + A_{54}B_{40} + A_{55}B_{50} + A_{56}B_{60} + A_{57}B_{70} \\
C_{60} &= \boldsymbol{B_{20}A_{40}} + \boldsymbol{B_{21}A_{50}} + \boldsymbol{B_{22}A_{60}} + \boldsymbol{B_{23}A_{70}} + A_{64}B_{40} + A_{65}B_{50} + A_{66}B_{60} + A_{67}B_{70} \\
C_{70} &= \boldsymbol{B_{30}A_{40}} + \boldsymbol{B_{31}A_{50}} + \boldsymbol{B_{32}A_{60}} + \boldsymbol{B_{33}A_{70}} + A_{74}B_{40} + A_{75}B_{50} + A_{76}B_{60} + A_{77}B_{70}
\end{aligned}
$$

$$
\begin{aligned}
C_{01} &= A_{00}B_{01} + A_{01}B_{11} + A_{02}B_{21} + A_{03}B_{31} + \boldsymbol{B_{40}A_{05}} + \boldsymbol{B_{41}A_{15}} + \boldsymbol{B_{42}A_{25}} + \boldsymbol{B_{43}A_{35}} \\
C_{11} &= A_{10}B_{01} + A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} + \boldsymbol{B_{50}A_{05}} + B_{51}A_{15} + \boldsymbol{B_{52}A_{25}} + \boldsymbol{B_{53}A_{35}} \\
C_{21} &= A_{20}B_{01} + A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} + \boldsymbol{B_{60}A_{05}} + \boldsymbol{B_{61}A_{15}} + \boldsymbol{B_{62}A_{25}} + \boldsymbol{B_{63}A_{35}} \\
C_{31} &= A_{30}B_{01} + A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} + \boldsymbol{B_{70}A_{05}} + \boldsymbol{B_{71}A_{15}} + \boldsymbol{B_{72}A_{25}} + \boldsymbol{B_{73}A_{35}} \\
C_{41} &= \boldsymbol{B_{00}A_{41}} + \boldsymbol{B_{01}A_{51}} + \boldsymbol{B_{02}A_{61}} + \boldsymbol{B_{03}A_{71}} + A_{44}B_{41} + A_{45}B_{51} + A_{46}B_{61} + A_{47}B_{71} \\
C_{51} &= \boldsymbol{B_{10}A_{41}} + B_{11}A_{51} + \boldsymbol{B_{12}A_{61}} + \boldsymbol{B_{13}A_{71}} + A_{54}B_{41} + A_{55}B_{51} + A_{56}B_{61} + A_{57}B_{71} \\
C_{61} &= \boldsymbol{B_{20}A_{41}} + \boldsymbol{B_{21}A_{51}} + \boldsymbol{B_{22}A_{61}} + \boldsymbol{B_{23}A_{71}} + A_{64}B_{41} + A_{65}B_{51} + A_{66}B_{61} + A_{67}B_{71} \\
C_{71} &= \boldsymbol{B_{30}A_{41}} + \boldsymbol{B_{31}A_{51}} + \boldsymbol{B_{32}A_{61}} + \boldsymbol{B_{33}A_{71}} + A_{74}B_{41} + A_{75}B_{51} + A_{76}B_{61} + A_{77}B_{71}
\end{aligned}
$$

$$
\begin{aligned}
C_{02} &= A_{00}B_{02} + A_{01}B_{12} + A_{02}B_{22} + A_{03}B_{32} + \boldsymbol{B_{40}A_{06}} + \boldsymbol{B_{41}A_{16}} + \boldsymbol{B_{42}A_{26}} + \boldsymbol{B_{43}A_{36}} \\
C_{12} &= A_{10}B_{02} + A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} + \boldsymbol{B_{50}A_{06}} + \boldsymbol{B_{51}A_{16}} + \boldsymbol{B_{52}A_{26}} + \boldsymbol{B_{53}A_{36}} \\
C_{22} &= A_{20}B_{02} + A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} + \boldsymbol{B_{60}A_{06}} + \boldsymbol{B_{61}A_{16}} + B_{62}A_{26} + \boldsymbol{B_{63}A_{36}} \\
C_{32} &= A_{30}B_{02} + A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} + \boldsymbol{B_{70}A_{06}} + \boldsymbol{B_{71}A_{16}} + \boldsymbol{B_{72}A_{26}} + \boldsymbol{B_{73}A_{36}} \\
C_{42} &= \boldsymbol{B_{00}A_{42}} + \boldsymbol{B_{01}A_{52}} + \boldsymbol{B_{02}A_{62}} + \boldsymbol{B_{03}A_{72}} + A_{44}B_{42} + A_{45}B_{52} + A_{46}B_{62} + A_{47}B_{72} \\
C_{52} &= \boldsymbol{B_{10}A_{42}} + \boldsymbol{B_{11}A_{52}} + \boldsymbol{B_{12}A_{62}} + \boldsymbol{B_{13}A_{72}} + A_{54}B_{42} + A_{55}B_{52} + A_{56}B_{62} + A_{57}B_{72} \\
C_{62} &= \boldsymbol{B_{20}A_{42}} + \boldsymbol{B_{21}A_{52}} + B_{22}A_{62} + \boldsymbol{B_{23}A_{72}} + A_{64}B_{42} + A_{65}B_{52} + A_{66}B_{62} + A_{67}B_{72} \\
C_{72} &= \boldsymbol{B_{30}A_{42}} + \boldsymbol{B_{31}A_{52}} + \boldsymbol{B_{32}A_{62}} + \boldsymbol{B_{33}A_{72}} + A_{74}B_{42} + A_{75}B_{52} + A_{76}B_{62} + A_{77}B_{72}
\end{aligned}
$$

$$
\begin{aligned}
C_{03} &= A_{00}B_{03} + A_{01}B_{13} + A_{02}B_{23} + A_{03}B_{33} + \boldsymbol{B_{40}A_{07}} + \boldsymbol{B_{41}A_{17}} + \boldsymbol{B_{42}A_{27}} + \boldsymbol{B_{43}A_{37}} \\
C_{13} &= A_{10}B_{03} + A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} + \boldsymbol{B_{50}A_{07}} + \boldsymbol{B_{51}A_{17}} + \boldsymbol{B_{52}A_{27}} + \boldsymbol{B_{53}A_{37}} \\
C_{23} &= A_{20}B_{03} + A_{21}B_{13} + A_{22}B_{23} + A_{23}B_{33} + \boldsymbol{B_{60}A_{07}} + \boldsymbol{B_{61}A_{17}} + \boldsymbol{B_{62}A_{27}} + \boldsymbol{B_{63}A_{37}} \\
C_{33} &= A_{30}B_{03} + A_{31}B_{13} + A_{32}B_{23} + A_{33}B_{33} + \boldsymbol{B_{70}A_{07}} + \boldsymbol{B_{71}A_{17}} + \boldsymbol{B_{72}A_{27}} + B_{73}A_{37} \\
C_{43} &= \boldsymbol{B_{00}A_{43}} + \boldsymbol{B_{01}A_{53}} + \boldsymbol{B_{02}A_{63}} + \boldsymbol{B_{03}A_{73}} + A_{44}B_{43} + A_{45}B_{53} + A_{46}B_{63} + A_{47}B_{73} \\
C_{53} &= \boldsymbol{B_{10}A_{43}} + \boldsymbol{B_{11}A_{53}} + \boldsymbol{B_{12}A_{63}} + \boldsymbol{B_{13}A_{73}} + A_{54}B_{43} + A_{55}B_{53} + A_{56}B_{63} + A_{57}B_{73} \\
C_{63} &= \boldsymbol{B_{20}A_{43}} + \boldsymbol{B_{21}A_{53}} + \boldsymbol{B_{22}A_{63}} + \boldsymbol{B_{23}A_{73}} + A_{64}B_{43} + A_{65}B_{53} + A_{66}B_{63} + A_{67}B_{73} \\
C_{73} &= \boldsymbol{B_{30}A_{43}} + \boldsymbol{B_{31}A_{53}} + \boldsymbol{B_{32}A_{63}} + B_{33}A_{73} + A_{74}B_{43} + A_{75}B_{53} + A_{76}B_{63} + A_{77}B_{73}
\end{aligned}
$$

$$
\begin{aligned}
C_{04} &= B_{04}A_{00} + \boldsymbol{B_{05}A_{10}} + \boldsymbol{B_{06}A_{20}} + \boldsymbol{B_{07}A_{30}} + A_{04}B_{44} + A_{05}B_{54} + A_{06}B_{64} + A_{07}B_{74} \\
C_{14} &= \boldsymbol{B_{14}A_{00}} + \boldsymbol{B_{15}A_{10}} + \boldsymbol{B_{16}A_{20}} + \boldsymbol{B_{17}A_{30}} + A_{14}B_{44} + A_{15}B_{54} + A_{16}B_{64} + A_{17}B_{74} \\
C_{24} &= \boldsymbol{B_{24}A_{00}} + \boldsymbol{B_{25}A_{10}} + \boldsymbol{B_{26}A_{20}} + \boldsymbol{B_{27}A_{30}} + A_{24}B_{44} + A_{25}B_{54} + A_{26}B_{64} + A_{27}B_{74} \\
C_{34} &= \boldsymbol{B_{34}A_{00}} + \boldsymbol{B_{35}A_{10}} + \boldsymbol{B_{36}A_{20}} + \boldsymbol{B_{37}A_{30}} + A_{34}B_{44} + A_{35}B_{54} + A_{36}B_{64} + A_{37}B_{74} \\
C_{44} &= A_{40}B_{04} + A_{41}B_{14} + A_{42}B_{24} + A_{43}B_{34} + B_{44}A_{44} + \boldsymbol{B_{45}A_{54}} + \boldsymbol{B_{46}A_{64}} + \boldsymbol{B_{47}A_{74}} \\
C_{54} &= A_{50}B_{04} + A_{51}B_{14} + A_{52}B_{24} + A_{53}B_{34} + \boldsymbol{B_{54}A_{44}} + \boldsymbol{B_{55}A_{54}} + \boldsymbol{B_{56}A_{64}} + \boldsymbol{B_{57}A_{74}} \\
C_{64} &= A_{60}B_{04} + A_{61}B_{14} + A_{62}B_{24} + A_{63}B_{34} + \boldsymbol{B_{64}A_{44}} + \boldsymbol{B_{65}A_{54}} + \boldsymbol{B_{66}A_{64}} + \boldsymbol{B_{67}A_{74}} \\
C_{74} &= A_{70}B_{04} + A_{71}B_{14} + A_{72}B_{24} + A_{73}B_{34} + \boldsymbol{B_{74}A_{44}} + \boldsymbol{B_{75}A_{54}} + \boldsymbol{B_{76}A_{64}} + \boldsymbol{B_{77}A_{74}}
\end{aligned}
$$

$$
\begin{aligned}
C_{05} &= \boldsymbol{B_{04}A_{01}} + \boldsymbol{B_{05}A_{11}} + \boldsymbol{B_{06}A_{21}} + \boldsymbol{B_{07}A_{31}} + A_{04}B_{45} + A_{05}B_{55} + A_{06}B_{65} + A_{07}B_{75} \\
C_{15} &= \boldsymbol{B_{14}A_{01}} + B_{15}A_{11} + \boldsymbol{B_{16}A_{21}} + \boldsymbol{B_{17}A_{31}} + A_{14}B_{45} + A_{15}B_{55} + A_{16}B_{65} + A_{17}B_{75} \\
C_{25} &= \boldsymbol{B_{24}A_{01}} + \boldsymbol{B_{25}A_{11}} + \boldsymbol{B_{26}A_{21}} + \boldsymbol{B_{27}A_{31}} + A_{24}B_{45} + A_{25}B_{55} + A_{26}B_{65} + A_{27}B_{75} \\
C_{35} &= \boldsymbol{B_{34}A_{01}} + \boldsymbol{B_{35}A_{11}} + \boldsymbol{B_{36}A_{21}} + \boldsymbol{B_{37}A_{31}} + A_{34}B_{45} + A_{35}B_{55} + A_{36}B_{65} + A_{37}B_{75} \\
C_{45} &= A_{40}B_{05} + A_{41}B_{15} + A_{42}B_{25} + A_{43}B_{35} + \boldsymbol{B_{44}A_{45}} + \boldsymbol{B_{45}A_{55}} + \boldsymbol{B_{46}A_{65}} + \boldsymbol{B_{47}A_{75}} \\
C_{55} &= A_{50}B_{05} + A_{51}B_{15} + A_{52}B_{25} + A_{53}B_{35} + \boldsymbol{B_{54}A_{45}} + B_{55}A_{55} + \boldsymbol{B_{56}A_{65}} + \boldsymbol{B_{57}A_{75}} \\
C_{65} &= A_{60}B_{05} + A_{61}B_{15} + A_{62}B_{25} + A_{63}B_{35} + \boldsymbol{B_{64}A_{45}} + \boldsymbol{B_{65}A_{55}} + \boldsymbol{B_{66}A_{65}} + \boldsymbol{B_{67}A_{75}} \\
C_{75} &= A_{70}B_{05} + A_{71}B_{15} + A_{72}B_{25} + A_{73}B_{35} + \boldsymbol{B_{74}A_{45}} + \boldsymbol{B_{75}A_{55}} + \boldsymbol{B_{76}A_{65}} + \boldsymbol{B_{77}A_{75}}
\end{aligned}
$$

$$
\begin{aligned}
C_{06} &= \boldsymbol{B_{04}A_{02}} + \boldsymbol{B_{05}A_{12}} + \boldsymbol{B_{06}A_{22}} + \boldsymbol{B_{07}A_{32}} + A_{04}B_{46} + A_{05}B_{56} + A_{06}B_{66} + A_{07}B_{76} \\
C_{16} &= \boldsymbol{B_{14}A_{02}} + \boldsymbol{B_{15}A_{12}} + \boldsymbol{B_{16}A_{22}} + \boldsymbol{B_{17}A_{32}} + A_{14}B_{46} + A_{15}B_{56} + A_{16}B_{66} + A_{17}B_{76} \\
C_{26} &= \boldsymbol{B_{24}A_{02}} + \boldsymbol{B_{25}A_{12}} + B_{26}A_{22} + \boldsymbol{B_{27}A_{32}} + A_{24}B_{46} + A_{25}B_{56} + A_{26}B_{66} + A_{27}B_{76} \\
C_{36} &= \boldsymbol{B_{34}A_{02}} + \boldsymbol{B_{35}A_{12}} + \boldsymbol{B_{36}A_{22}} + \boldsymbol{B_{37}A_{32}} + A_{34}B_{46} + A_{35}B_{56} + A_{36}B_{66} + A_{37}B_{76} \\
C_{46} &= A_{40}B_{06} + A_{41}B_{16} + A_{42}B_{26} + A_{43}B_{36} + \boldsymbol{B_{44}A_{46}} + \boldsymbol{B_{45}A_{56}} + \boldsymbol{B_{46}A_{66}} + \boldsymbol{B_{47}A_{76}} \\
C_{56} &= A_{50}B_{06} + A_{51}B_{16} + A_{52}B_{26} + A_{53}B_{36} + \boldsymbol{B_{54}A_{46}} + \boldsymbol{B_{55}A_{56}} + \boldsymbol{B_{56}A_{66}} + \boldsymbol{B_{57}A_{76}} \\
C_{66} &= A_{60}B_{06} + A_{61}B_{16} + A_{62}B_{26} + A_{63}B_{36} + \boldsymbol{B_{64}A_{46}} + \boldsymbol{B_{65}A_{56}} + B_{66}A_{66} + \boldsymbol{B_{67}A_{76}} \\
C_{76} &= A_{70}B_{06} + A_{71}B_{16} + A_{72}B_{26} + A_{73}B_{36} + \boldsymbol{B_{74}A_{46}} + \boldsymbol{B_{75}A_{56}} + \boldsymbol{B_{76}A_{66}} + \boldsymbol{B_{77}A_{76}}
\end{aligned}
$$

$$
\begin{aligned}
C_{07} &= \boldsymbol{B_{04}A_{03}} + \boldsymbol{B_{05}A_{13}} + \boldsymbol{B_{06}A_{23}} + \boldsymbol{B_{07}A_{33}} + A_{04}B_{47} + A_{05}B_{57} + A_{06}B_{67} + A_{07}B_{77} \\
C_{17} &= \boldsymbol{B_{14}A_{03}} + \boldsymbol{B_{15}A_{13}} + \boldsymbol{B_{16}A_{23}} + \boldsymbol{B_{17}A_{33}} + A_{14}B_{47} + A_{15}B_{57} + A_{16}B_{67} + A_{17}B_{77} \\
C_{27} &= \boldsymbol{B_{24}A_{03}} + \boldsymbol{B_{25}A_{13}} + \boldsymbol{B_{26}A_{23}} + \boldsymbol{B_{27}A_{33}} + A_{24}B_{47} + A_{25}B_{57} + A_{26}B_{67} + A_{27}B_{77} \\
C_{37} &= \boldsymbol{B_{34}A_{03}} + \boldsymbol{B_{35}A_{13}} + \boldsymbol{B_{36}A_{23}} + B_{37}A_{33} + A_{34}B_{47} + A_{35}B_{57} + A_{36}B_{67} + A_{37}B_{77} \\
C_{47} &= A_{40}B_{07} + A_{41}B_{17} + A_{42}B_{27} + A_{43}B_{37} + \boldsymbol{B_{44}A_{47}} + \boldsymbol{B_{45}A_{57}} + \boldsymbol{B_{46}A_{67}} + \boldsymbol{B_{47}A_{77}} \\
C_{57} &= A_{50}B_{07} + A_{51}B_{17} + A_{52}B_{27} + A_{53}B_{37} + \boldsymbol{B_{54}A_{47}} + \boldsymbol{B_{55}A_{57}} + \boldsymbol{B_{56}A_{67}} + \boldsymbol{B_{57}A_{77}} \\
C_{67} &= A_{60}B_{07} + A_{61}B_{17} + A_{62}B_{27} + A_{63}B_{37} + \boldsymbol{B_{64}A_{47}} + \boldsymbol{B_{65}A_{57}} + \boldsymbol{B_{66}A_{67}} + \boldsymbol{B_{67}A_{77}} \\
C_{77} &= A_{70}B_{07} + A_{71}B_{17} + A_{72}B_{27} + A_{73}B_{37} + \boldsymbol{B_{74}A_{47}} + \boldsymbol{B_{75}A_{57}} + \boldsymbol{B_{76}A_{67}} + B_{77}A_{77}
\end{aligned}
$$

THE 8-DIM × PRODUCT (ALL CAYLEY-DICKSON ALGEBRAS):

$$\begin{aligned}
C_{00} &= A_{00}B_{00} + B_{10}A_{01} + B_{20}A_{02} + \boldsymbol{A_{12}B_{21}} + B_{40}A_{04} + \boldsymbol{A_{14}B_{41}} + \boldsymbol{A_{24}B_{42}} + \boldsymbol{B_{52}A_{25}}\\
C_{10} &= B_{00}A_{10} + A_{11}B_{10} + \boldsymbol{A_{02}B_{30}} + \boldsymbol{B_{31}A_{12}} + \boldsymbol{A_{04}B_{50}} + \boldsymbol{B_{51}A_{14}} + \boldsymbol{B_{42}A_{34}} + \boldsymbol{A_{35}B_{52}}\\
C_{20} &= B_{00}A_{20} + \boldsymbol{A_{30}B_{01}} + A_{22}B_{20} + B_{30}A_{23} + \boldsymbol{A_{04}B_{60}} + \boldsymbol{B_{70}A_{05}} + \boldsymbol{B_{62}A_{24}} + \boldsymbol{A_{34}B_{63}}\\
C_{30} &= \boldsymbol{A_{20}B_{10}} + \boldsymbol{B_{11}A_{30}} + B_{20}A_{32} + A_{33}B_{30} + \boldsymbol{B_{60}A_{14}} + \boldsymbol{A_{15}B_{70}} + \boldsymbol{A_{24}B_{72}} + \boldsymbol{B_{73}A_{34}}\\
C_{40} &= B_{00}A_{40} + \boldsymbol{A_{50}B_{01}} + \boldsymbol{A_{60}B_{02}} + \boldsymbol{B_{12}A_{61}} + A_{44}B_{40} + B_{50}A_{45} + B_{60}A_{46} + \boldsymbol{A_{56}B_{61}}\\
C_{50} &= \boldsymbol{A_{40}B_{10}} + \boldsymbol{B_{11}A_{50}} + \boldsymbol{B_{02}A_{70}} + \boldsymbol{A_{71}B_{12}} + B_{40}A_{54} + A_{55}B_{50} + \boldsymbol{A_{46}B_{70}} + \boldsymbol{B_{71}A_{56}}\\
C_{60} &= \boldsymbol{A_{40}B_{20}} + \boldsymbol{B_{30}A_{41}} + \boldsymbol{B_{22}A_{60}} + \boldsymbol{A_{70}B_{23}} + B_{40}A_{64} + \boldsymbol{A_{74}B_{41}} + A_{66}B_{60} + B_{70}A_{67}\\
C_{70} &= \boldsymbol{B_{20}A_{50}} + \boldsymbol{A_{51}B_{30}} + \boldsymbol{A_{60}B_{32}} + \boldsymbol{B_{33}A_{70}} + \boldsymbol{A_{64}B_{50}} + \boldsymbol{B_{51}A_{74}} + B_{60}A_{76} + A_{77}B_{70}
\end{aligned}$$

$$\begin{aligned}
C_{01} &= B_{01}A_{00} + A_{01}B_{11} + \boldsymbol{A_{03}B_{20}} + \boldsymbol{B_{21}A_{13}} + \boldsymbol{A_{05}B_{40}} + \boldsymbol{B_{41}A_{15}} + \boldsymbol{B_{43}A_{24}} + \boldsymbol{A_{25}B_{53}}\\
C_{11} &= A_{10}B_{01} + B_{11}A_{11} + \boldsymbol{B_{30}A_{03}} + A_{13}B_{31} + \boldsymbol{B_{50}A_{05}} + A_{15}B_{51} + \boldsymbol{A_{34}B_{43}} + \boldsymbol{B_{53}A_{35}}\\
C_{21} &= \boldsymbol{A_{21}B_{00}} + \boldsymbol{B_{01}A_{31}} + B_{21}A_{22} + A_{23}B_{31} + \boldsymbol{B_{61}A_{04}} + \boldsymbol{A_{05}B_{71}} + \boldsymbol{A_{25}B_{62}} + \boldsymbol{B_{63}A_{35}}\\
C_{31} &= \boldsymbol{B_{10}A_{21}} + A_{31}B_{11} + A_{32}B_{21} + B_{31}A_{33} + \boldsymbol{A_{14}B_{61}} + \boldsymbol{B_{71}A_{15}} + \boldsymbol{B_{72}A_{25}} + \boldsymbol{A_{35}B_{73}}\\
C_{41} &= \boldsymbol{A_{41}B_{00}} + \boldsymbol{B_{01}A_{51}} + \boldsymbol{B_{03}A_{60}} + \boldsymbol{A_{61}B_{13}} + B_{41}A_{44} + A_{45}B_{51} + \boldsymbol{A_{47}B_{60}} + \boldsymbol{B_{61}A_{57}}\\
C_{51} &= \boldsymbol{B_{10}A_{41}} + A_{51}B_{11} + \boldsymbol{A_{70}B_{03}} + \boldsymbol{B_{13}A_{71}} + A_{54}B_{41} + B_{51}A_{55} + \boldsymbol{B_{70}A_{47}} + A_{57}B_{71}\\
C_{61} &= \boldsymbol{B_{21}A_{40}} + \boldsymbol{A_{41}B_{31}} + \boldsymbol{A_{61}B_{22}} + \boldsymbol{B_{23}A_{71}} + \boldsymbol{A_{65}B_{40}} + \boldsymbol{B_{41}A_{75}} + B_{61}A_{66} + A_{67}B_{71}\\
C_{71} &= \boldsymbol{A_{50}B_{21}} + \boldsymbol{B_{31}A_{51}} + \boldsymbol{B_{32}A_{61}} + \boldsymbol{A_{71}B_{33}} + \boldsymbol{B_{50}A_{65}} + A_{75}B_{51} + A_{76}B_{61} + B_{71}A_{77}
\end{aligned}$$

$$\begin{aligned}
C_{02} &= B_{02}A_{00} + \boldsymbol{A_{10}B_{03}} + A_{02}B_{22} + B_{32}A_{03} + \boldsymbol{A_{06}B_{40}} + \boldsymbol{B_{50}A_{07}} + \boldsymbol{B_{42}A_{26}} + \boldsymbol{A_{36}B_{43}}\\
C_{12} &= \boldsymbol{A_{00}B_{12}} + \boldsymbol{B_{13}A_{10}} + B_{22}A_{12} + A_{13}B_{32} + \boldsymbol{B_{40}A_{16}} + \boldsymbol{A_{17}B_{50}} + \boldsymbol{A_{26}B_{52}} + \boldsymbol{B_{53}A_{36}}\\
C_{22} &= A_{20}B_{02} + B_{12}A_{21} + B_{22}A_{22} + \boldsymbol{A_{32}B_{23}} + \boldsymbol{B_{60}A_{06}} + \boldsymbol{A_{16}B_{61}} + A_{26}B_{62} + B_{72}A_{27}\\
C_{32} &= B_{02}A_{30} + A_{31}B_{12} + \boldsymbol{A_{22}B_{32}} + \boldsymbol{B_{33}A_{32}} + \boldsymbol{A_{06}B_{70}} + \boldsymbol{B_{71}A_{16}} + B_{62}A_{36} + A_{37}B_{72}\\
C_{42} &= \boldsymbol{A_{42}B_{00}} + \boldsymbol{B_{10}A_{43}} + \boldsymbol{B_{02}A_{62}} + \boldsymbol{A_{72}B_{03}} + B_{42}A_{44} + \boldsymbol{A_{54}B_{43}} + A_{46}B_{62} + B_{72}A_{47}\\
C_{52} &= \boldsymbol{B_{00}A_{52}} + \boldsymbol{A_{53}B_{10}} + \boldsymbol{A_{62}B_{12}} + \boldsymbol{B_{13}A_{72}} + \boldsymbol{A_{44}B_{52}} + \boldsymbol{B_{53}A_{54}} + B_{62}A_{56} + A_{57}B_{72}\\
C_{62} &= \boldsymbol{B_{20}A_{42}} + \boldsymbol{A_{52}B_{21}} + A_{62}B_{22} + B_{32}A_{63} + A_{64}B_{42} + B_{52}A_{65} + B_{62}A_{66} + \boldsymbol{A_{76}B_{63}}\\
C_{72} &= \boldsymbol{A_{42}B_{30}} + \boldsymbol{B_{31}A_{52}} + B_{22}A_{72} + A_{73}B_{32} + B_{42}A_{74} + A_{75}B_{52} + \boldsymbol{A_{66}B_{72}} + \boldsymbol{B_{73}A_{76}}
\end{aligned}$$

$$\begin{aligned}
C_{03} &= \boldsymbol{A_{01}B_{02}} + \boldsymbol{B_{03}A_{11}} + B_{23}A_{02} + A_{03}B_{33} + \boldsymbol{B_{41}A_{06}} + \boldsymbol{A_{07}B_{51}} + \boldsymbol{A_{27}B_{42}} + \boldsymbol{B_{43}A_{37}}\\
C_{13} &= \boldsymbol{B_{12}A_{01}} + A_{11}B_{13} + A_{12}B_{23} + B_{33}A_{13} + \boldsymbol{A_{16}B_{41}} + \boldsymbol{B_{51}A_{17}} + \boldsymbol{B_{52}A_{27}} + \boldsymbol{A_{37}B_{53}}\\
C_{23} &= B_{03}A_{20} + A_{21}B_{13} + \boldsymbol{A_{23}B_{22}} + \boldsymbol{B_{23}A_{33}} + \boldsymbol{A_{07}B_{60}} + \boldsymbol{B_{61}A_{17}} + B_{63}A_{26} + A_{27}B_{73}\\
C_{33} &= A_{30}B_{03} + B_{13}A_{31} + \boldsymbol{B_{32}A_{23}} + A_{33}B_{33} + \boldsymbol{B_{70}A_{07}} + \boldsymbol{A_{17}B_{71}} + A_{36}B_{63} + B_{73}A_{37}\\
C_{43} &= \boldsymbol{B_{01}A_{42}} + \boldsymbol{A_{43}B_{11}} + \boldsymbol{A_{63}B_{02}} + \boldsymbol{B_{03}A_{73}} + \boldsymbol{A_{45}B_{42}} + \boldsymbol{B_{43}A_{55}} + B_{63}A_{46} + A_{47}B_{73}\\
C_{53} &= \boldsymbol{A_{52}B_{01}} + \boldsymbol{B_{11}A_{53}} + \boldsymbol{B_{12}A_{63}} + \boldsymbol{A_{73}B_{13}} + \boldsymbol{B_{52}A_{45}} + A_{55}B_{53} + A_{56}B_{63} + B_{73}A_{57}\\
C_{63} &= \boldsymbol{A_{43}B_{20}} + \boldsymbol{B_{21}A_{53}} + B_{23}A_{62} + A_{63}B_{33} + B_{43}A_{64} + A_{65}B_{53} + \boldsymbol{A_{67}B_{62}} + \boldsymbol{B_{63}A_{77}}\\
C_{73} &= \boldsymbol{B_{30}A_{43}} + \boldsymbol{A_{53}B_{31}} + A_{72}B_{23} + B_{33}A_{73} + A_{74}B_{43} + B_{53}A_{75} + \boldsymbol{B_{72}A_{67}} + A_{77}B_{73}
\end{aligned}$$

$$\begin{aligned}
C_{04} &= B_{04}A_{00} + \boldsymbol{A_{10}B_{05}} + \boldsymbol{A_{20}B_{06}} + \boldsymbol{B_{16}A_{21}} + A_{04}B_{44} + B_{54}A_{05} + B_{64}A_{06} + \boldsymbol{A_{16}B_{65}}\\
C_{14} &= \boldsymbol{A_{00}B_{14}} + \boldsymbol{B_{15}A_{10}} + \boldsymbol{B_{06}A_{30}} + \boldsymbol{A_{31}B_{16}} + B_{44}A_{14} + A_{15}B_{54} + \boldsymbol{A_{06}B_{74}} + \boldsymbol{B_{75}A_{16}}\\
C_{24} &= \boldsymbol{A_{00}B_{24}} + \boldsymbol{B_{34}A_{01}} + \boldsymbol{B_{26}A_{20}} + \boldsymbol{A_{30}B_{27}} + B_{44}A_{24} + \boldsymbol{A_{34}B_{45}} + A_{26}B_{64} + B_{74}A_{27}\\
C_{34} &= \boldsymbol{B_{24}A_{10}} + \boldsymbol{A_{11}B_{34}} + \boldsymbol{A_{20}B_{36}} + \boldsymbol{B_{37}A_{30}} + \boldsymbol{A_{24}B_{54}} + \boldsymbol{B_{55}A_{34}} + B_{64}A_{36} + A_{37}B_{74}\\
C_{44} &= A_{40}B_{04} + B_{14}A_{41} + B_{24}A_{42} + \boldsymbol{A_{52}B_{25}} + B_{44}A_{44} + \boldsymbol{A_{54}B_{45}} + \boldsymbol{A_{64}B_{46}} + \boldsymbol{B_{56}A_{65}}\\
C_{54} &= B_{04}A_{50} + A_{51}B_{14} + \boldsymbol{A_{42}B_{34}} + \boldsymbol{B_{35}A_{52}} + \boldsymbol{A_{44}B_{54}} + \boldsymbol{B_{55}A_{54}} + \boldsymbol{B_{46}A_{74}} + \boldsymbol{A_{75}B_{56}}\\
C_{64} &= B_{04}A_{60} + \boldsymbol{A_{70}B_{05}} + A_{62}B_{24} + B_{34}A_{63} + \boldsymbol{A_{44}B_{64}} + \boldsymbol{B_{74}A_{45}} + \boldsymbol{B_{66}A_{64}} + \boldsymbol{A_{74}B_{67}}\\
C_{74} &= \boldsymbol{A_{60}B_{14}} + \boldsymbol{B_{15}A_{70}} + B_{24}A_{72} + A_{73}B_{34} + \boldsymbol{B_{64}A_{54}} + \boldsymbol{A_{55}B_{74}} + \boldsymbol{A_{64}B_{76}} + \boldsymbol{B_{77}A_{74}}
\end{aligned}$$

$$\begin{aligned}
C_{05} &= \boldsymbol{A_{01}B_{04}} + \boldsymbol{B_{05}A_{11}} + \boldsymbol{B_{07}A_{20}} + \boldsymbol{A_{21}B_{17}} + B_{45}A_{04} + A_{05}B_{55} + \boldsymbol{A_{07}B_{64}} + \boldsymbol{B_{65}A_{17}}\\
C_{15} &= \boldsymbol{B_{14}A_{01}} + A_{11}B_{15} + \boldsymbol{A_{30}B_{07}} + \boldsymbol{B_{17}A_{31}} + A_{14}B_{45} + B_{55}A_{15} + \boldsymbol{B_{74}A_{07}} + A_{17}B_{75}\\
C_{25} &= \boldsymbol{B_{25}A_{00}} + \boldsymbol{A_{01}B_{35}} + \boldsymbol{A_{21}B_{26}} + \boldsymbol{B_{27}A_{31}} + \boldsymbol{A_{25}B_{44}} + \boldsymbol{B_{45}A_{35}} + B_{65}A_{26} + A_{27}B_{75}\\
C_{35} &= \boldsymbol{A_{10}B_{25}} + \boldsymbol{B_{35}A_{11}} + \boldsymbol{B_{36}A_{21}} + \boldsymbol{A_{31}B_{37}} + \boldsymbol{B_{54}A_{25}} + A_{35}B_{55} + A_{36}B_{65} + B_{75}A_{37}\\
C_{45} &= B_{05}A_{40} + A_{41}B_{15} + \boldsymbol{A_{43}B_{24}} + \boldsymbol{B_{25}A_{53}} + \boldsymbol{A_{45}B_{44}} + \boldsymbol{B_{45}A_{55}} + \boldsymbol{B_{47}A_{64}} + \boldsymbol{A_{65}B_{57}}\\
C_{55} &= A_{50}B_{05} + B_{15}A_{51} + \boldsymbol{B_{34}A_{43}} + A_{53}B_{35} + \boldsymbol{B_{54}A_{45}} + A_{55}B_{55} + \boldsymbol{A_{74}B_{47}} + \boldsymbol{B_{57}A_{75}}\\
C_{65} &= \boldsymbol{A_{61}B_{04}} + \boldsymbol{B_{05}A_{71}} + B_{25}A_{62} + A_{63}B_{35} + \boldsymbol{B_{65}A_{44}} + \boldsymbol{A_{45}B_{75}} + \boldsymbol{A_{65}B_{66}} + \boldsymbol{B_{67}A_{75}}\\
C_{75} &= \boldsymbol{B_{14}A_{61}} + A_{71}B_{15} + A_{72}B_{25} + B_{35}A_{73} + \boldsymbol{A_{54}B_{65}} + \boldsymbol{B_{75}A_{55}} + \boldsymbol{B_{76}A_{65}} + \boldsymbol{A_{75}B_{77}}
\end{aligned}$$

$$\begin{aligned}
C_{06} &= \boldsymbol{A_{02}B_{04}} + \boldsymbol{B_{14}A_{03}} + \boldsymbol{B_{06}A_{22}} + \boldsymbol{A_{32}B_{07}} + B_{46}A_{04} + \boldsymbol{A_{14}B_{47}} + A_{06}B_{66} + B_{76}A_{07}\\
C_{16} &= \boldsymbol{B_{04}A_{12}} + \boldsymbol{A_{13}B_{14}} + \boldsymbol{A_{22}B_{16}} + \boldsymbol{B_{17}A_{32}} + \boldsymbol{A_{04}B_{56}} + \boldsymbol{B_{57}A_{14}} + B_{66}A_{16} + A_{17}B_{76}\\
C_{26} &= \boldsymbol{B_{24}A_{02}} + \boldsymbol{A_{12}B_{25}} + A_{22}B_{26} + B_{36}A_{23} + A_{24}B_{46} + B_{56}A_{25} + B_{66}A_{26} + \boldsymbol{A_{36}B_{67}}\\
C_{36} &= \boldsymbol{A_{02}B_{34}} + \boldsymbol{B_{35}A_{12}} + B_{26}A_{32} + A_{33}B_{36} + B_{46}A_{34} + A_{35}B_{56} + \boldsymbol{A_{26}B_{76}} + \boldsymbol{B_{77}A_{36}}\\
C_{46} &= B_{06}A_{40} + \boldsymbol{A_{50}B_{07}} + A_{42}B_{26} + B_{36}A_{43} + \boldsymbol{A_{46}B_{44}} + \boldsymbol{B_{54}A_{47}} + \boldsymbol{B_{46}A_{66}} + \boldsymbol{A_{76}B_{47}}\\
C_{56} &= \boldsymbol{A_{40}B_{16}} + \boldsymbol{B_{17}A_{50}} + B_{26}A_{52} + A_{53}B_{36} + \boldsymbol{B_{44}A_{56}} + \boldsymbol{A_{57}B_{54}} + \boldsymbol{A_{66}B_{56}} + \boldsymbol{B_{57}A_{76}}\\
C_{66} &= A_{60}B_{06} + B_{16}A_{61} + B_{26}A_{62} + \boldsymbol{A_{72}B_{27}} + \boldsymbol{B_{64}A_{46}} + \boldsymbol{A_{56}B_{65}} + A_{66}B_{66} + B_{76}A_{67}\\
C_{76} &= B_{06}A_{70} + A_{71}B_{16} + \boldsymbol{A_{62}B_{36}} + \boldsymbol{B_{37}A_{72}} + \boldsymbol{A_{46}B_{74}} + \boldsymbol{B_{75}A_{56}} + B_{66}A_{76} + A_{77}B_{76}
\end{aligned}$$

$$\begin{aligned}
C_{07} &= \boldsymbol{B_{05}A_{02}} + \boldsymbol{A_{03}B_{15}} + \boldsymbol{A_{23}B_{06}} + \boldsymbol{B_{07}A_{33}} + \boldsymbol{A_{05}B_{46}} + \boldsymbol{B_{47}A_{15}} + B_{67}A_{06} + A_{07}B_{77}\\
C_{17} &= \boldsymbol{A_{12}B_{05}} + \boldsymbol{B_{15}A_{13}} + \boldsymbol{B_{16}A_{23}} + \boldsymbol{A_{33}B_{17}} + \boldsymbol{B_{56}A_{05}} + A_{15}B_{57} + A_{16}B_{67} + B_{77}A_{17}\\
C_{27} &= \boldsymbol{A_{03}B_{24}} + \boldsymbol{B_{25}A_{13}} + B_{27}A_{22} + A_{23}B_{37} + B_{47}A_{24} + A_{25}B_{57} + \boldsymbol{A_{27}B_{66}} + \boldsymbol{B_{67}A_{37}}\\
C_{37} &= \boldsymbol{B_{34}A_{03}} + \boldsymbol{A_{13}B_{35}} + A_{32}B_{27} + B_{37}A_{33} + A_{34}B_{47} + B_{57}A_{35} + \boldsymbol{B_{76}A_{27}} + A_{37}B_{77}\\
C_{47} &= \boldsymbol{A_{41}B_{06}} + \boldsymbol{B_{07}A_{51}} + B_{27}A_{42} + A_{43}B_{37} + \boldsymbol{B_{45}A_{46}} + \boldsymbol{A_{47}B_{55}} + \boldsymbol{A_{67}B_{46}} + \boldsymbol{B_{47}A_{77}}\\
C_{57} &= \boldsymbol{B_{16}A_{41}} + A_{51}B_{17} + A_{52}B_{27} + B_{37}A_{53} + \boldsymbol{A_{56}B_{45}} + \boldsymbol{B_{55}A_{57}} + \boldsymbol{B_{56}A_{67}} + \boldsymbol{A_{77}B_{57}}\\
C_{67} &= B_{07}A_{60} + A_{61}B_{17} + \boldsymbol{A_{63}B_{26}} + \boldsymbol{B_{27}A_{73}} + \boldsymbol{A_{47}B_{64}} + \boldsymbol{B_{65}A_{57}} + B_{67}A_{66} + A_{67}B_{77}\\
C_{77} &= A_{70}B_{07} + B_{17}A_{71} + \boldsymbol{B_{36}A_{63}} + A_{73}B_{37} + \boldsymbol{B_{74}A_{47}} + \boldsymbol{A_{57}B_{75}} + A_{76}B_{67} + B_{77}A_{77}
\end{aligned}$$

APPENDIX B (PAGE A)— THE 32 ELEMENT BASIS MATRICES FOR $\mathbb{H}\times\mathbb{O}$

$$e_0 = \begin{pmatrix} +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&+1 \end{pmatrix}
\quad e_1 = \begin{pmatrix} .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&+1&. \end{pmatrix}
\quad e_2 = \begin{pmatrix} .&.&-1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \end{pmatrix}
\quad e_3 = \begin{pmatrix} .&.&.&-1&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&-1&.&.&. \end{pmatrix}$$

$$e_4 = \begin{pmatrix} .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&-1 \\ +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \end{pmatrix}
\quad e_5 = \begin{pmatrix} .&.&.&.&.&-1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \\ .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&-1&.&.&.&.&. \end{pmatrix}
\quad e_6 = \begin{pmatrix} .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \end{pmatrix}
\quad e_7 = \begin{pmatrix} .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \end{pmatrix}$$

$$\boldsymbol{i}\times e_0 = \begin{pmatrix} .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \end{pmatrix}
\quad \boldsymbol{i}\times e_1 = \begin{pmatrix} -1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&+1 \end{pmatrix}
\quad \boldsymbol{i}\times e_2 = \begin{pmatrix} .&.&.&+1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&+1&.&.&. \end{pmatrix}
\quad \boldsymbol{i}\times e_3 = \begin{pmatrix} .&.&-1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \end{pmatrix}$$

$$\boldsymbol{i}\times e_4 = \begin{pmatrix} .&.&.&.&.&+1&.&. \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \\ .&+1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&-1&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{i}\times e_5 = \begin{pmatrix} .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&+1 \\ +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \end{pmatrix}
\quad \boldsymbol{i}\times e_6 = \begin{pmatrix} .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{i}\times e_7 = \begin{pmatrix} .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \end{pmatrix}$$

$$\boldsymbol{j}\times e_0 = \begin{pmatrix} .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{j}\times e_1 = \begin{pmatrix} .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{j}\times e_2 = \begin{pmatrix} .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&+1&. \\ .&-1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{j}\times e_3 = \begin{pmatrix} .&.&.&.&-1&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \\ +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \end{pmatrix}$$

$$\boldsymbol{j}\times e_4 = \begin{pmatrix} .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&-1&.&.&. \end{pmatrix}
\quad \boldsymbol{j}\times e_5 = \begin{pmatrix} .&.&-1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&+1&.&. \end{pmatrix}
\quad \boldsymbol{j}\times e_6 = \begin{pmatrix} .&-1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&-1&. \end{pmatrix}
\quad \boldsymbol{j}\times e_7 = \begin{pmatrix} +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&.&.&-1 \end{pmatrix}$$

$$\boldsymbol{k}\times e_0 = \begin{pmatrix} .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{k}\times e_1 = \begin{pmatrix} .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \end{pmatrix}
\quad \boldsymbol{k}\times e_2 = \begin{pmatrix} .&.&.&.&+1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \\ -1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \end{pmatrix}
\quad \boldsymbol{k}\times e_3 = \begin{pmatrix} .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&-1&. \\ .&-1&.&.&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&+1&.&.&.&.&. \end{pmatrix}$$

$$\boldsymbol{k}\times e_4 = \begin{pmatrix} .&.&-1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ -1&.&.&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&-1&.&.&. \\ .&.&.&.&.&+1&.&. \end{pmatrix}
\quad \boldsymbol{k}\times e_5 = \begin{pmatrix} .&.&.&+1&.&.&.&. \\ .&.&+1&.&.&.&.&. \\ .&+1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&.&.&.&.&+1 \\ .&.&.&.&.&.&+1&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \end{pmatrix}
\quad \boldsymbol{k}\times e_6 = \begin{pmatrix} +1&.&.&.&.&.&.&. \\ .&-1&.&.&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&+1&.&.&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&-1&.&. \\ .&.&.&.&.&.&-1&. \\ .&.&.&.&.&.&.&+1 \end{pmatrix}
\quad \boldsymbol{k}\times e_7 = \begin{pmatrix} .&+1&.&.&.&.&.&. \\ +1&.&.&.&.&.&.&. \\ .&.&.&-1&.&.&.&. \\ .&.&-1&.&.&.&.&. \\ .&.&.&.&.&+1&.&. \\ .&.&.&.&+1&.&.&. \\ .&.&.&.&.&.&.&-1 \\ .&.&.&.&.&.&-1&. \end{pmatrix}$$

APPENDIX B (PAGE B)— THE REMAINING 32 ELEMENT BASIS MATRICES FOR $M_{[\times]}(8,\mathbb{R})$

$$\boldsymbol{l}\times e_0 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_1 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_2 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_3 = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_4 = \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_5 = \begin{pmatrix} \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_6 = \begin{pmatrix} \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times e_7 = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_0) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_1) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_2) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_3) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_4) = \begin{pmatrix} \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_5) = \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_6) = \begin{pmatrix} -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{i}\times e_7) = \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_0) = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_1) = \begin{pmatrix} \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_2) = \begin{pmatrix} \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_3) = \begin{pmatrix} \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_4) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_5) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_6) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{j}\times e_7) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_0) = \begin{pmatrix} \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_1) = \begin{pmatrix} +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_2) = \begin{pmatrix} \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_3) = \begin{pmatrix} \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_4) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_5) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_6) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & +1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

$$\boldsymbol{l}\times(\boldsymbol{k}\times e_7) = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +1 \\ \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\ \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot \\ +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & +1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

C.1

APPENDIX C

THE × PRODUCT IN THE HEXPENTAQUATERNION ALGEBRA $(\mathbb{X}_n, \times)$

$\times$	E	I_A	J_A	K_A	I_R	J_R	K_R	I_M	J_M	K_M	I_L	J_L	K_L	I_Z	J_Z	K_Z
E	E	I_A	J_A	K_A	I_R	J_R	K_R	I_M	J_M	K_M	I_L	J_L	K_L	I_Z	J_Z	K_Z
I_A	I_A	E	$-K_L$	$-J_R$	$-J_M$	$-K_A$	J_Z	I_Z	$-I_R$	$-I_L$	$-K_M$	K_Z	$-J_A$	I_M	K_R	J_L
J_A	J_A	K_L	E	$-I_Z$	K_R	$-I_M$	I_R	$-J_R$	J_Z	$-J_L$	$-K_Z$	$-K_M$	I_A	$-K_A$	J_M	$-I_L$
K_A	K_A	J_R	$-I_Z$	E	$-J_L$	I_A	K_M	$-K_L$	$-K_Z$	K_R	J_Z	$-I_R$	$-I_M$	$-J_A$	I_L	$-J_M$
I_R	I_R	J_M	K_R	$-J_L$	$-E$	$-K_Z$	$-J_A$	$-I_L$	$-I_A$	I_Z	I_M	K_A	J_Z	$-K_M$	$-K_L$	J_R
J_R	J_R	K_A	I_M	$-I_A$	$-K_Z$	$-E$	$-I_L$	$-J_A$	$-J_L$	J_Z	K_R	J_M	I_Z	$-K_L$	$-K_M$	I_R
K_R	K_R	$-J_Z$	I_R	$-K_M$	$-J_A$	I_L	$-E$	K_Z	K_L	K_A	$-J_R$	I_Z	$-J_M$	$-J_L$	I_A	$-I_M$
I_M	I_M	I_Z	J_R	K_L	$-I_L$	J_A	$-K_Z$	E	$-K_M$	$-J_M$	$-I_R$	$-J_Z$	K_A	I_A	$-J_L$	$-K_R$
J_M	J_M	I_R	J_Z	$-K_Z$	I_A	$-J_L$	K_L	$-K_M$	E	$-I_M$	$-I_Z$	$-J_R$	K_R	$-I_L$	J_A	$-K_A$
K_M	K_M	I_L	J_L	$-K_R$	$-I_Z$	$-J_Z$	$-K_A$	$-J_M$	$-I_M$	E	I_A	J_A	K_Z	$-I_R$	$-J_R$	K_L
I_L	I_L	K_M	K_Z	$-J_Z$	I_M	$-K_R$	J_R	$-I_R$	I_Z	$-I_A$	$-E$	$-K_L$	J_L	$-J_M$	K_A	$-J_A$
J_L	J_L	$-K_Z$	K_M	$-I_R$	K_A	J_M	I_Z	J_Z	$-J_R$	$-J_A$	K_L	$-E$	$-I_L$	$-K_R$	$-I_M$	I_A
K_L	K_L	J_A	$-I_A$	I_M	J_Z	I_Z	$-J_M$	$-K_A$	K_R	$-K_Z$	$-J_L$	I_L	$-E$	$-J_R$	$-I_R$	K_M
I_Z	I_Z	I_M	$-K_A$	$-J_A$	K_M	$-K_L$	$-J_L$	I_A	I_L	I_R	J_M	$-K_R$	$-J_R$	E	$-K_Z$	$-J_Z$
J_Z	J_Z	$-K_R$	J_M	$-I_L$	$-K_L$	K_M	$-I_A$	J_L	J_A	J_R	$-K_A$	I_M	$-I_R$	$-K_Z$	E	$-I_Z$
K_Z	K_Z	$-J_L$	I_L	$-J_M$	J_R	I_R	I_M	K_R	$-K_A$	$-K_L$	J_A	$-I_A$	$-K_M$	$-J_Z$	$-I_Z$	E

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[1] Sometimes the Cayley-Dickson process is generalized with an extra field parameter, μ, thus one finds, $(A,B)(C,D)_I = (AC \pm \mu DB^*, A^*D + CB)$, or $(A,B)(C,D)_{II} = (AC \pm \mu D^*B, DA + BC^*)$. A more modern "Conway-Smith process" introduces the form, $(A,B)(C,D) = (AC - (BD^*)^*, (B^*C^*)^* + (B^*(A^*((B^{-1})^*D^*)^*)^*)^*)$, to be used when $B \neq 0$, and replaced by, $(A,B)(C,D) = (AC, A^*D)$, if $B = 0$. This process produces the same four normed division algebras as the Cayley-Dickson Process, where the formula is equivalent to construction (I), but then results in a different set of uniquely distinct "Conway-Smith algebras" beyond the octonions. The main advantage of this modification is that one recovers the law that the product of two sums of n squares a is sum of n squares. see eq.68 of [WS], and pg.79 of [CS] for their (II) form.

[2] The $\vdash$ and $\dashv$ symbols indicate division from the left and right, $B\backslash A = \dfrac{A}{\vdash B} = B^{-1}A$ and $A/B = \dfrac{A}{B \dashv} = AB^{-1}$.

[3] The "alternating complex numbers" were first discovered in 1849 by James Cockle and called "coquaternions." Other names appearing in the literature for these same 4-dimensional numbers are the "split-quaternions" and the "hyperbolic quaternions." The symbol $\ddot{\mathbb{H}}$ we've selected for them reflects the "split" character, hence the two dots $\ddot{}$. However, in this paper we prefer to refer to their "alternating" character.

General solutions to linear problems in quaternion variables.

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We present a method to solve linear quaternion problems using the right and left hand forms of the quaternion. This technique was first introduced in our previous paper on hexpentaquaternions, but parts of the method were only sketched out there. In this paper, we give all the remaining specific details for solving linear quaternion equations and their corresponding linear systems. The quaternion expansion of the determinant, and quaternion expansion of the adjoint matrix, for any real four square matrix, are also given, and provide the keys to the general solutions. A previous unsolved quaternion problem is then solved to illustrate the effectiveness of the two-hand quaternions.

. INTRODUCTION .

In a previous paper[1] [PJ2], we introduced a method for solving linear quaternion equations, by including both right handed and left handed quaternions in the same algebra. Prior high art followed Hamilton's method of writing the quaternion as the sum of scalar and vector, $q = Sq + Vq$, in the method of solutions, while working entirely in the right hand quaternion system alone[1]. This method had a number of drawbacks, making solutions to many problems difficult, cumbersome, and non-intuitive. The alternative low art method of reckoning with the components, $q = q_0 + q_1 i + q_2 j + q_3 k$, was often resorted to in tackling many types of problems. In both of these methods the quaternion had to be broken up into parts just to reason with these non-abelian numbers. In our two-hand quaternion method, we don't break the quaternion up, instead we employ the two operations of conjugation * and hand transformation $'$ to modify expressions and manipulate them into useful alternative forms. This algebra possesses the usual associative and distributive laws of ordinary algebra, while incorporating special commutative laws that enable us to move things around and effect solutions. The first special commutative law is, $qB = B'\hat{q}$, where, $q, B \in \mathbb{H}_R$ and $B' \in \mathbb{H}_L$. This allows us to induce a permutation of parameters in cases where the initial pair of quaternions in the binary product are of the same hand. The second special commutative law is, $AB' = B'A$, where, $A \in \mathbb{H}_R$ and $B' \in \mathbb{H}_L$. This tells us that right handed and left handed quaternions commute with each other. The third commutative principle we make use of is that scalars commute with all numbers, so we recognise those special combinations of quaternions that are scalar, $AA^* \in \mathbb{R}$ and $A + A^* \in \mathbb{R}$, etc., and use this fact to permute factors in expressions.

When $A, B \in \mathbb{H}_R$, then, $A^*, B^* \in \mathbb{H}_R$, and $A', B', A'^*, B'^* \in \mathbb{H}_L$, and we recall some useful rules,

$(A^*)^* = A$	$(A')' = A$	$A^{-1} = A^*/\lvert A\rvert^2$
$AB \neq BA$	$A'B' \neq B'A'$	$AB' = B'A$
$(AB)^* = B^*A^*$	$(AB)' = B'A'$	$(A'B')' = BA$
$(AB')^* = B'^*A^* = A^*B'^*$	$(AB')' = BA' = A'B$	$(A^*)' = (A')^*$
$AA^*, A + A^* \in \mathbb{R}$	$AA^* = A^*A = A'A'^* = \lvert A\rvert^2$	$A + A^* = A' + A'^*$

These are essentially all the special rules[2] we need in working out solutions, and together with the usual rules common to ordinary algebra, we can solve these linear problems. Hamilton's quaternions traditionally use the symbol, $\mathbb{H}$, but they are right handed only. Since we're working with both hands, we add a subscript and write, $\mathbb{H}_R$, for the usual right handed quaternions, and, $\mathbb{H}_L$, for the corresponding left handed quaternions. But, when a number contains a mix of both hands, it's referred to as an hexpe(two-hand quaternion, hexpentaquaternion) number, $\mathbb{X}_n$. We also sometimes use the terms " hand " and " handed " interchangeably, prefering the shorter word to the longer, but using the latter whenever distinction needs to be made between the " right hand side " parameter and the " right handed " parameter, or in other situations where greater clarification helps. Having said this, all problems considered in this paper are initially stated in the right hand quaternion algebra, and the left hand is introduced to facilitate reckoning.

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I. LINEAR EQUATIONS IN ONE VARIABLE.

$$\begin{aligned} A_1 q B_1 &= C & (1)\\ A_1 q B_1 + A_2 q B_2 &= C & (2)\\ A_1 q B_1 + A_2 q B_2 + A_3 q B_3 &= C & (3)\\ &\vdots \\ A_1 q B_1 + A_2 q B_2 + \cdots + A_n q B_n &= C & (4) \end{aligned}$$

With all parameters being right hand quaternions, $q, A_k, B_k, C \in \mathbb{H}_R\,; k = 1,..,n$, the above linear equations in one unknown, q, are now to be solved; first, for $n \leq 3$, then another way for all n. The first equation (1) is straightforward, and is easily solved within right hand algebra. However, we illustrate the two-hand method for completeness,

$$\begin{aligned} A_1 q B_1 &= C & (1)\\ A_1 B_1' \hat{q} &= \hat{C} & (5)\\ A_1^* A_1 B_1' \hat{q} &= A_1^* \hat{C} & (6)\\ |A_1|^2 B_1'^* B_1' \hat{q} &= B_1'^* A_1^* \hat{C} & (7)\\ |A_1|^2 |B_1|^2 \hat{q} &= A_1^* B_1'^* \hat{C} & (8)\\ \hat{q} &= \frac{1}{|A_1|^2 |B_1|^2} \cdot A_1^* B_1'^* \hat{C} & (9)\\ q &= \frac{1}{|A_1|^2 |B_1|^2} \cdot A_1^* C B_1^* & (10) \end{aligned}$$

Of course , we could write this as, $q = A_1^{-1} C B_1^{-1}$, which is the form readily recognised from the obvious one-hand solution method. But, a few points are illustrated here. The commutative law, $qB_1 = B_1' \hat{q}$, is used to induce the first permutation. The right handed quaternion, B_1, moves from the R-H-S over to the L-H-S of the q parameter and changes it's handedness from right-handed to left-handed; $B_1 \rightarrow B_1' \in \mathbb{H}_L$. Meanwhile, the fixed quaternion parameter takes on a hat, $\hat{q}$, to indicate that it is being used as the pivot about which parameter movements are made. This is the commutative law for pivots, given in our previous paper[1] [PJ2]. The inhomogeneous parameter is also promoted to pivot status, so it gets hatted by a caret also; $C \rightarrow \hat{C}$. This procedure transforms the problem from the one-hand quaternion algebra into the equivalent two-hand quaternion algebra format. We then proceed in the usual manner to find appropriate factors to reduce the known quaternions to scalar numbers on the L-H-S of the equation. While, on the R-H-S of the equation we apply the second commutative law that allows us to permute right-handed with left-handed quaternions, $B_1'^* A_1^* = A_1^* B_1'^*$. Finally, once we have the unknown by itself on the L-H-S of the equation, we use the commutative law for pivots again, $B_1'^* \hat{C} = C B_1^*$, allowing us to remove the caret ˆ from the quaternions, and present the solution entirely in the right hand quaternion algebra, $\mathbb{H}_R$.

The second equation (2) is substantially more involved, but the basic ideas are all illustrated above in the solution to the first. Now we have to contend with variable parameters that are mixes of right and left quaternions, i.e. general hexpe numbers from the set, $\mathbb{X}_n$; and need to reduce these two-hand quaternions to scalars. We'd proceed as follows,

$$A_1 q B_1 + A_2 q B_2 = C \qquad (2)$$

$$A_1 B_1' \hat{q} + A_2 B_2' \hat{q} = \hat{C} \qquad (11)$$

$$(A_1 B_1' + A_2 B_2') \hat{q} = \hat{C} \qquad (12)$$

$$\hat{q} = (A_1 B_1' + A_2 B_2')^{-1} \hat{C} \qquad (13)$$

We have our unknown by itself on one side, but to complete this solution we need to find that inverse factor. Now we recall the fact that every hexpe number can be expressed as the sum of pair products, where each pair has one right and one left hand quaternion, therefore we can always write this inverse factor in the form,

$$(A_1 B_1' + A_2 B_2')^{-1} = (P_1 Q_1' + P_2 Q_2' + \ldots + P_m Q_m') \qquad \text{where, } P_k, Q_k \in \mathbb{H}_R \qquad (14)$$

whence,

$$\begin{aligned} \hat{q} &= (P_1 Q_1' + P_2 Q_2' + \cdots + P_m Q_m')\hat{C} && (15) \\ &= P_1 Q_1' \hat{C} + P_2 Q_2' \hat{C} + \cdots + P_m Q_m' \hat{C} && (16) \end{aligned}$$

$$q = P_1 C Q_1 + P_2 C Q_2 + \cdots + P_m C Q_m \qquad (17)$$

and we have our solution. What we need, however, is to find the $\{P_k, Q_k\}$ in terms of the the $\{A_k, B_k\}$(our previous paper[1] [PJ2] left out these specific details, so in this paper we now complete the method, showing all working art.). Let us then introduce a few new parameters, $h, g, F_k;\ k = 1, 2$; with,

$$\begin{aligned} h &= A_1 B_1' + A_2 B_2' && (18) \\ g &= A_1^* F_1' + A_2^* F_2' \qquad \text{where, } F_1, F_2 \in \mathbb{H}_R && (19) \\ \therefore \quad gh &= (A_1^* F_1' + A_2^* F_2')(A_1 B_1' + A_2 B_2') && (20) \\ &= A_1^* F_1' A_1 B_1' + A_2^* F_2' A_2 B_2' + A_1^* F_1' A_2 B_2' + A_2^* F_2' A_1 B_1' && (21) \\ &= A_1^* A_1 F_1' B_1' + A_2^* A_2 F_2' B_2' + A_1^* A_2 F_1' B_2' + A_2^* A_1 F_2' B_1' && (22) \\ &= |A_1|^2 F_1' B_1' + |A_2|^2 F_2' B_2' + A_1^* A_2 F_1' B_2' + A_2^* A_1 F_2' B_1' && (23) \end{aligned}$$

and now pick, F_1, F_2, so that, $F_2' B_1' = F_1' B_2'$, and therefore, $F_2' = F_1'(B_2'/B_1')$, and we then have, $gh \in \mathbb{H}_L$, i.e.,

$$gh = |A_1|^2 F_1' B_1' + |A_2|^2 F_1'(B_2'/B_1') B_2' + (A_1^* A_2 + A_2^* A_1) F_1' B_2' \quad \in \mathbb{H}_L \qquad (24)$$

$$\text{since,} \qquad |A_1|^2, |A_2|^2, (A_1^* A_2 + A_2^* A_1) = (A_1^* A_2 + (A_1^* A_2)^*) \quad \in \mathbb{R} \qquad (25)$$

Now that gh is a one-hand quaternion, we know its inverse, $(gh)^{-1} = (gh)^*/|gh|^2$, we can write, $h^{-1} = (gh)^* g/|gh|^2$. That is, we can solve, $h\hat{q} = \hat{C} \ \rightarrow \ h^{-1} h \hat{q} = h^{-1}\hat{C} \ \rightarrow \ 1\hat{q} = h^{-1}\hat{C}$, and this h^{-1} is a *left side inverse* of h. But, since all $h \in \mathbb{X}_n$ are representable by 4×4 matrices over reals, the left side inverse is also the right side inverse. The expression, (B_2'/B_1'), can be written, $(B_2' B_1'^*)/|B_1|^2$, and, to remove the dividing term, we then pick $F_1' = +|B_1|$. So,

$$\begin{aligned} g &= A_1^* |B_1| + A_2^* B_2' B_1'^* / |B_1| && (26) \\ gh &= |A_1|^2 |B_1| B_1' + (|A_2|^2/|B_1|) B_2' B_1'^* B_2' + (A_1^* A_2 + A_2^* A_1)|B_1| B_2' \quad \in \mathbb{H}_L && (27) \end{aligned}$$

$$\begin{aligned} (gh)^* g &= |A_1|^2 A_1^* |B_1|^2 B_1'^* + |A_2|^2 A_1^* B_2'^* B_1' B_2'^* + (A_1^* A_2 + A_2^* A_1) A_1^* B_2'^* |B_1|^2 && (28) \\ &+ |A_1|^2 A_2^* B_1'^* B_2' B_1'^* + |A_2|^2 A_2^* B_2'^* |B_2|^2 + (A_1^* A_2 + A_2^* A_1) A_2^* B_1'^* |B_2|^2 \end{aligned}$$

$$\begin{aligned} &= A_1^* A_1 A_1^* B_1'^* B_1' B_1'^* + A_2^* A_2 A_1^* B_2'^* B_1' B_2'^* + A_1^* A_2 A_1^* B_2'^* B_1' B_1'^* + A_2^* A_1 A_1^* B_2'^* B_1' B_1'^* && (29) \\ &+ A_1^* A_1 A_2^* B_1'^* B_2' B_1'^* + A_2^* A_2 A_2^* B_2'^* B_2' B_2'^* + A_1^* A_2 A_2^* B_1'^* B_2' B_2'^* + A_2^* A_1 A_2^* B_1'^* B_2' B_2'^* \end{aligned}$$

Permuting various factors and moving terms around allows us to re-write this more symmetrically[3],

$$\begin{aligned} (gh)^* g = \; & A_1^*\, A_1\, A_1^* \;\; B_1'^*\, B_1'\, B_1'^* \\ + \; & A_1^*\, A_1\, A_2^* \;\; B_1'^*\, B_1'\, B_2'^* \\ + \; & A_1^*\, A_2\, A_1^* \;\; B_1'^*\, B_1'\, B_2'^* \\ + \; & A_1^*\, A_2\, A_2^* \;\; B_1'^*\, B_2'\, B_2'^* \\ & && (30) \\ + \; & A_2^*\, A_1\, A_1^* \;\; B_1'^*\, B_2'\, B_1'^* \\ + \; & A_2^*\, A_1\, A_2^* \;\; B_2'^*\, B_2'\, B_1'^* \\ + \; & A_2^*\, A_2\, A_1^* \;\; B_2'^*\, B_1'\, B_2'^* \\ + \; & A_2^*\, A_2\, A_2^* \;\; B_2'^*\, B_2'\, B_2'^* \end{aligned}$$

Thus, in the eqns (14)-(17), apart from a scalar factor in the denominator, the P_k have the form $A_.^* A_. A_.^*$, while the Q_k have the form $B_.^* B_. B_.^*$, where the subscript dot $_.$ is a placeholder that represents the appropriate index. The solution therefore has the form,

$$q = \frac{\sum A_.^* A_. A_.^* C B_.^* B_. B_.^*}{|gh|^2} \qquad (31)$$

We can write[4]

$$|gh|^2 = (gh)^*(gh) = (h^*g^*)(gh) = h^*(g^*g)h \tag{32}$$

Since, $g \in \mathbb{X}_n$ is not usually a one-hand quaternion, $g^*g \notin \mathbb{R}$. So, we can't just replace $h^*(g^*g)h$ with $(h^*h)(g^*g)$, we have to work out the result of the transformation $h^*(\ )h$ on the value of g^*g , instead.

$$\begin{aligned} g^*g &= (A_1|B_1| + A_2B_1'B_2'^*/|B_1|)(A_1^*|B_1| + A_2^*B_2'B_1'^*/|B_1|) && (33)\\ &= |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 + A_1A_2^*B_2'B_1'^* + A_2A_1^*B_1'B_2'^* && (34)\end{aligned}$$

$$\begin{aligned} h^*h &= (A_1^*B_1'^* + A_2^*B_2'^*)(A_1B_1' + A_2B_2') && (35)\\ &= |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 + A_1^*A_2B_1'^*B_2' + A_2^*A_1B_2'^*B_1' && (36)\end{aligned}$$

$$\begin{aligned} h^*(g^*g)h &= (\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )h^*h + h^*(\ A_1A_2^*B_2'B_1'^* + A_2A_1^*B_1'B_2'^*\ )h && (37)\\ &= (\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )^2 + (\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )(A_1^*A_2B_1'^*B_2' + A_2^*A_1B_2'^*B_1') && (38)\\ &+ (A_1^*B_1'^* + A_2^*B_2'^*)(\ A_1A_2^*B_2'B_1'^* + A_2A_1^*B_1'B_2'^*\ )(A_1B_1' + A_2B_2') \\ &= (\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )^2 \\ &+ (\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )((A_1^*A_2) + (A_1^*A_2)^*)((B_2B_1^*)' + (B_2B_1^*)'^*) && (39)\\ &+ |B_1|^2|B_2|^2((A_1^*A_2)^2 + (A_1^*A_2)^{2*}) + |A_1|^2|A_2|^2(((B_2B_1^*)^2)' + ((B_2B_1^*)^2)'^*)\end{aligned}$$

We then use the fact that expressions which resolve to scalar values can swap the hands throughout, e.g. $A + A^* = A' + A'^*$, etc.. to write, $(B_2B_1^*)' + (B_2B_1^*)'^* = (B_2B_1^*) + (B_2B_1^*)^*$, etc.. and re-express this whole result in right hand quaternion parameters. Then we use the general result, $(A + A^*)^2 - 2|A|^2 = (A)^2 + (A^2)^*$, to simplify again, and if we let, $2\alpha = A_1^*A_2 + (A_1^*A_2)^*$, and $2\beta = B_1^*B_2 + (B_1^*B_2)^*$, we can write this as[5],

$$|gh|^2 = (|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2$$

The inverse factor from eq(13) is then,

$$(A_1B_1' + A_2B_2')^{-1} = \left(\frac{\begin{gathered}(|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2)(A_1^*B_1'^* + A_2^*B_2'^*) + |A_2|^2A_1^*(B_2^*B_1B_2^*)' \\ +|A_1|^2A_2^*(B_1^*B_2B_1^*)' + |B_2|^2A_2^*A_1A_2^*B_1'^* + |B_1|^2A_1^*A_2A_1^*B_2'^*\end{gathered}}{\begin{gathered}(|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + \\ 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2\end{gathered}} \right) \tag{40}$$

and the final solution for q is,

$$q = \left(\frac{\begin{gathered}(|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2)(A_1^*CB_1^* + A_2^*CB_2^*) + |A_2|^2A_1^*CB_2^*B_1B_2^* \\ +|A_1|^2A_2^*CB_1^*B_2B_1^* + |B_2|^2A_2^*A_1A_2^*CB_1^* + |B_1|^2A_1^*A_2A_1^*CB_2^*\end{gathered}}{\begin{gathered}(|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + \\ 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2\end{gathered}} \right) \tag{41}$$

where, $2\alpha = A_1^*A_2 + (A_1^*A_2)^*$, $2\beta = B_1^*B_2 + (B_1^*B_2)^*$

We can alternatively solve the equation (2) from the right side instead.

$$A_1 q B_1 + A_2 q B_2 = C \tag{2}$$
$$\hat{q} A_1' B_1 + \hat{q} A_2' B_2 = \hat{C} \tag{42}$$
$$\hat{q}(A_1' B_1 + A_2' B_2) = \hat{C} \tag{43}$$
$$\hat{q} = \hat{C}(A_1' B_1 + A_2' B_2)^{-1} \tag{44}$$

Here we use the pivot commutation law on the other side, $A_1 q = \hat{q} A_1'$, and the A_k become left-handed this time, while the B_k remain right-handed. Proceeding similarily, we introduce parameters, $h^\diamond, g^\diamond, F_k; k = 1, 2;$, with,

$$h^\diamond = A_1' B_1 + A_2' B_2 \tag{45}$$
$$g^\diamond = A_1'^* F_1 + A_2'^* F_2 \qquad \text{where, } F_1, F_2 \in \mathbb{H}_R \tag{46}$$
$$\therefore \quad h^\diamond g^\diamond = |A_1|^2 B_1 F_1 + |A_2|^2 B_2 F_2 + A_1' A_2'^* B_1 F_2 + A_2' A_1'^* B_2 F_1 \tag{47}$$

then pick, F_1, F_2, so that, $B_1 F_2 = B_2 F_1$, and therefore, $F_2 = (B_1 \backslash B_2) F_1$, and we then have, $h^\diamond g^\diamond \in \mathbb{H}_R$, with usual inverse, $(h^\diamond g^\diamond)^{-1} = (h^\diamond g^\diamond)^* / |h^\diamond g^\diamond|^2$, and, after working out, we find that the inverse factor from eq(44) is then,

$$(A_1' B_1 + A_2' B_2)^{-1} = \left(\frac{\begin{array}{l} (|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2)(A_1'^* B_1^* + A_2'^* B_2^*) + |A_2|^2 A_1'^* (B_2^* B_1 B_2^*) \\ + |A_1|^2 A_2'^* (B_1^* B_2 B_1^*) + |B_2|^2 (A_2^* A_1 A_2^*)' B_1^* + |B_1|^2 (A_1^* A_2 A_1^*)' B_2^* \end{array}}{\begin{array}{l} (|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + \\ 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2 \end{array}} \right) \tag{48}$$

$$\text{where, } 2\alpha = A_1^* A_2 + (A_1^* A_2)^* \ , \ \ 2\beta = B_1^* B_2 + (B_1^* B_2)^*$$

Apart from the hand swapping, the intermediate steps of the procedure are very similar to that before, and the final solution is the same; Evaluating, $\hat{q} = \hat{C}(h^\diamond)^{-1}$, by applying the pivot commutation law again, $\hat{C} A_1'^* \to A_1^* C$, etc..results in the same expression for q given in eqn (41). Note that (48) is the hand transform of (40), i.e.

$$(A_1' B_1 + A_2' B_2)^{-1} = ((A_1 B_1')' + (A_2 B_2')')^{-1} = ((A_1 B_1' + A_2 B_2')')^{-1} = ((A_1 B_1' + A_2 B_2')^{-1})' \tag{49}$$

so that, $(h^\diamond)^{-1} = (h^{-1})'$.

We should point out that although we speak of splitting the quaternion representation into *two* states, q and $\hat{q}$, in order to effect solutions, that hatted state is not identical between the two commutes, $qB = B'\hat{q}$ and $Aq = \hat{q}A'$. This is not usually an issue since we don't mix these two methods together when working out solutions—the whole object, after all, is to be able to apply the distributive law in the end, which requires the unknown be on one side only. By tipping the hat to the right, $\hat{q} \to q\hat{}$, or to the left, $\hat{q} \to \hat{}q$, so we can write, $qB = B'q\hat{}$, and, $Aq = \hat{}qA'$, instead, we could indicate the distinction between the two pivot states. In the matrix representation of this two-hand quaternion algebra, the right tipped hat, $q\hat{}$, would be a column vector, while the left tipped hat, $\hat{}q$, a row vector. However, this extra distinction is usually unnecessary, and the top hat format, $\hat{q}$, is adequate. We also won't show more solution methods working from the right side of the unknown, since the proccess is practically the same as that when working from the left side, and the intermediate expressions can always be inferred by a simple hand transformation, as illustrated in (49). So, for the rest of this paper we work from the left side and use only the law, $qB = B'\hat{q}$.

With hat tipping, we could symbolize this equivalence of left and right side methods by,

$$h^{-1}C\hat{} \quad \hat{=} \quad \hat{}C(h^\diamond)^{-1} \quad = \quad \hat{}C(h^{-1})' \tag{50}$$

where the special equal sign, $\hat{=}$, is taken to mean " transpose and equate ": i.e. $(\ h^{-1}C\hat{}\ )^T \ = \ \hat{}C(h^{-1})'$.

In the special case, where, $A_2 = 1, B_1 = 1$, and, $A_1 = X, B_2 = Y$, we have, $h^{-1} = (X + Y')^{-1}$, and we obtain,

$$h^{-1} \;=\; \frac{(|X|^2+|Y|^2)(X^*+Y'^*)+X^*(Y^*Y^*)'+|X|^2(Y)'+|Y|^2X+X^*X^*Y'^*}{(|X|^2-|Y|^2)^2\;+(\;|X|^2+|Y|^2\;)(X+X^*)(Y+Y^*)+|Y|^2(X+X^*)^2+|X|^2(Y+Y^*)^2} \tag{51}$$

$$\therefore \quad h^{-1}\hat{C} \;=\; \frac{(|X|^2+|Y|^2)(X^*C+CY^*)+X^*CY^*Y^*+|X|^2CY+|Y|^2XC+X^*X^*CY^*}{(|X|^2-|Y|^2)^2\;+(\;|X|^2+|Y|^2\;)(X+X^*)(Y+Y^*)+|Y|^2(X+X^*)^2+|X|^2(Y+Y^*)^2} \tag{52}$$

This special case solution was previously given in eqns (72) and (73) of our Quatro-Quaternion paper[2] [PJ3], where we obtained it by modifying yet another previous result for the general hexpe number inverse found in our earlier Hexpentaquaternion paper[1] [PJ2], and in the earlier paper we used a rather lengthy method of matrix algebra to obtain the inverse. So, it's useful to compare the results, especially since the formulas take slightly different forms. We reproduce the previous two eqns below, as (QQ-72) and (QQ-73), for convenience. The inhomogeneous parameter, C, above, is equivalent to that paper's, Z, parameter below. The pivot can be written, $\hat{C} \equiv C \cdot \hat{1}$, in two-hand algebra, but the dot $\cdot$ appearing here has no special significance, it just references the same usual multiplication operation.

$$\texttt{Given,}\quad X, Y, Z \in \mathbb{H}_R;\;\; h = X + Y' \in \mathbb{X}_n,\;\; \mathcal{S}X = X_0,\;\; \mathcal{S}Y = Y_0,\;\; \mathcal{V}X = X - X_0,\;\; \mathcal{V}Y = Y - Y_0,\;\texttt{etc..}$$

$$h^{-1} \cdot Z \cdot \hat{1} = \tag{QQ-72}$$

$$\frac{(\,(X_0+Y_0)^2(X^*Z+ZY^*)+|X-X_0|^2(X^*Z+ZY)+|Y-Y_0|^2(XZ+ZY^*)+2(X_0+Y_0)(X-X_0)Z(Y-Y_0)\,)}{((X_0+Y_0)^2+|X-X_0|^2+|Y-Y_0|^2)^2-4|X-X_0|^2|Y-Y_0|^2}$$

$$= \frac{(\,(\mathcal{S}(X+Y))^2(X^*Z+ZY^*)+|\mathcal{V}X|^2(X^*Z+ZY)+|\mathcal{V}Y|^2(XZ+ZY^*)+2(\mathcal{S}(X+Y))(\mathcal{V}X)Z(\mathcal{V}Y)\,)}{((\mathcal{S}(X+Y))^2+|\mathcal{V}X|^2+|\mathcal{V}Y|^2)^2-4|\mathcal{V}X|^2|\mathcal{V}Y|^2} \tag{QQ-73}$$

By writing, $X = X_0 + (X - X_0)$ and $Y = Y_0 + (Y - Y_0)$, and using these to replace parameters in (52), we can easily demonstrate the equivalence to the result in (QQ-72); recall, $X^* = X_0 - (X - X_0)$, $(X - X_0)^2 = -|X - X_0|^2$, etc... thus, we may permute square forms, like $C(Y - Y_0)^2 = (Y - Y_0)^2C$, and so move them about anywhere in product expressions, since the square of a pure quaternion is just a scalar. It is probably a lot easier to start with (52) and manipulate this formula to derive (QQ-72), rather than the other way around. This formula is repeated again in (QQ-73) using a mix of old and new notation. We borrow two of Hamilton's original symbols, $\mathcal{S}X$ and $\mathcal{V}X$, to replace the somewhat more cumbersome forms, X_0 and $X - X_0$, of the modern notation. But, we prefer to keep the modern conjugate, X^*, and norm, $|X|^2$, when writing these expressions[6]. The old scalar and vector forms do stand out more on the page, rendering the formulas slightly more memorable, however. So, we use them on occasion. Sticking strictly to Hamilton's original notation, where S, V, K, N, T, U, are used for the scalar, vector, conjugate, norm, tensor, and versor parts of a quaternion, the solution in eqn (QQ-73) above would now be written,

$$= \frac{(\,(\mathcal{S}(X+Y))^2((\mathcal{K}X)Z+Z\mathcal{K}Y)+(\mathcal{N}\mathcal{V}X)((\mathcal{K}X)Z+ZY)+(\mathcal{N}\mathcal{V}Y)(XZ+Z\mathcal{K}Y)+2(\mathcal{S}(X+Y))(\mathcal{V}X)Z(\mathcal{V}Y)\,)}{((\mathcal{S}(X+Y))^2+(\mathcal{N}\mathcal{V}X)+(\mathcal{N}\mathcal{V}Y))^2-4(\mathcal{N}\mathcal{V}X)(\mathcal{N}\mathcal{V}Y)} \tag{QQ-73'}$$

One can become comfortable with any system of symbols, and with frequent use these all come to seem natural. However, this format uses many more letters and takes up significantly more space on the page. Letters are also best reserved for variables, whenever possible, since it just makes reading a little easier. So, although this form may have some advantages today when developing computer algorithms for symbolic manipulation of quaternions, we won't generally make use of this older format in writing papers to present the subject. There are also several ways to arrange the terms in these formulas, that give different looking expressions, it being not readily apparent that they are the same. The best forms are really dependent on the nature of the application problem being studied with these formulas. However, it helps to be able to see, at a glance, that the denominator evaluates to a scalar, so we tend to favor expressing things wherever possible so that each term looks like a scalar there, and one does not have to mentally compute an expression to verify it results in just a real number. But, the best numerator format is application dependent; seeing vectors helps with physical applications, while whole quaternions is better for algebra.

For example, our solution in (40)-(41), which has six quaternion terms in the numerator sum, can be further reduced to four terms, by combining appropriate quaternion expressions into scalars. In our previous formulas, the "norm", e.g. $|A|^2$, was the only scalar allowed in the numerator. But, if we relax this requirement, and also allow scalars formed with combinations of different quaternions, e.g. $A_i^* A_j + A_j^* A_i \in \mathbb{R}$, etc.., then, we can write,

$$(A_1 B_1' + A_2 B_2')^{-1} = \left(\frac{\begin{array}{l}(|A_1|^2|B_1|^2 - |B_2|^2|A_2|^2)(A_1^* B_1'^* - A_2^* B_2'^*) \\ + 2(|A_2|^2\beta + |B_1|^2\alpha)A_1^* B_2'^* + 2(|A_1|^2\beta + |B_2|^2\alpha)A_2^* B_1'^*\end{array}}{\begin{array}{l}(|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + \\ 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2\end{array}} \right) \tag{53}$$

and the final solution for q is,

$$q = \left(\frac{\begin{array}{l}(|A_1|^2|B_1|^2 - |B_2|^2|A_2|^2)(A_1^* C B_1^* - A_2^* C B_2^*) \\ + 2(|A_2|^2\beta + |B_1|^2\alpha)A_1^* C B_2^* + 2(|A_1|^2\beta + |B_2|^2\alpha)A_2^* C B_1^*\end{array}}{\begin{array}{l}(|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + \\ 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2\end{array}} \right) \tag{54}$$

$$\text{where, } 2\alpha = A_1^* A_2 + (A_1^* A_2)^*\ ,\ \ 2\beta = B_1^* B_2 + (B_1^* B_2)^*$$

All these formulas remain invariant under index exchange, i.e. $1 \to 2,\ 2 \to 1$. This is expected, since the problem definition is unchanged by re-arrangement of the AqB terms in the linear sum. This fact gives us a quick way to check results, and identify calculation errors, in what is sometimes a long and tedious manipulation of symbols.

The solution,

$$A_1 q B_1 + A_2 q B_2 = C \tag{2}$$

$$\therefore \quad q = \frac{w_1}{\lambda} A_1^* C B_1^* + \frac{w_2}{\lambda} A_1^* C B_2^* + \frac{w_3}{\lambda} A_2^* C B_1^* + \frac{w_4}{\lambda} A_2^* C B_2^* \qquad \text{where, } w_k, \lambda \in \mathbb{R}$$

has four terms that are just the various combinations of the ordered pair (A_j, B_k), with, $j, k = 1, 2$.

$$\begin{aligned} w_1 &= (|A_1|^2|B_1|^2 - |B_2|^2|A_2|^2) \ = \ -w_4 \\ w_2 &= 2(|A_2|^2\beta + |B_1|^2\alpha) \\ w_3 &= 2(|A_1|^2\beta + |B_2|^2\alpha) \\ \lambda &= (|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2)^2 + 4(\ |A_1|^2|B_1|^2 + |A_2|^2|B_2|^2\ )\alpha\beta + 4|B_1|^2|B_2|^2\alpha^2 + 4|A_1|^2|A_2|^2\beta^2 \end{aligned}$$

Although there are originally $2 \times 2 \times 2 = 2^3 = 8$ terms, in the numerator, with the form $A_.^* A_. A_.^* B_.^* B_. B_.^*$, several terms combine to give scalars, ultimately reducing these to the number of irreducible quaternion terms, which in this case is $2 + 2 \cdot 1 + 0 = 4$. In general, when there are n terms $A_. q B_.$ in the initial linear equation, the solution will have *at most* n^3 simple terms—i.e. with unit weight factors—of the form $A_.^* A_. A_.^* q B_.^* B_. B_.^*$, in the numerator, and this sum can then be re-expressed in $(2n^3 - 3n^2 + 4n)/3, \forall\, n \geq 1$, irreducible quaternion terms. The proof of this is found later in this paper. The number of irreducible quaternion terms with arbitrary scalar coefficients, which this initial numerator can be reduced to; for $n = 2$, is 4; for $n = 3$, is 13; for $n = 4$, is 32; and for $n = 5$, is 65.

In equation (3) we have,

$$\begin{aligned} A_1 q B_1 + A_2 q B_2 + A_3 q B_3 &= C && (3) \\ A_1 B_1' \hat{q} + A_2 B_2' \hat{q} + A_3 B_3' \hat{q} &= \hat{C} && (55) \\ (A_1 B_1' + A_2 B_2' + A_3 B_3')\hat{q} &= \hat{C} && (56) \\ \hat{q} &= (A_1 B_1' + A_2 B_2' + A_3 B_3')^{-1} \hat{C} && (57) \end{aligned}$$

We proceed similarly to the second equation and introduce new parameters, $h, g, F_k; k = 1,2,3$; with,

$$\begin{aligned} h &= A_1 B_1' + A_2 B_2' + A_3 B_3' && (58) \\ g &= A_1^* F_1' + A_2^* F_2' + A_3^* F_3' \qquad \text{where, } F_1, F_2, F_3 \in \mathbb{H}_R && (59) \\ \therefore \quad gh &= (A_1^* F_1' + A_2^* F_2' + A_3^* F_3')(A_1 B_1' + A_2 B_2' + A_3 B_3') && (60) \\ &= |A_1|^2 F_1' B_1' + |A_2|^2 F_2' B_2' + |A_3|^2 F_3' B_3' && (61) \\ &+ A_1^* A_2 F_1' B_2' + A_2^* A_1 F_2' B_1' \\ &+ A_2^* A_3 F_2' B_3' + A_3^* A_2 F_3' B_2' \\ &+ A_3^* A_1 F_3' B_1' + A_1^* A_3 F_1' B_3' \end{aligned}$$

Let's pick, F_1', F_2', F_3', and introduce the scalars, $\alpha_1, \alpha_2, \alpha_3$, where,

$$\begin{array}{lll} F_1' B_2' = F_2' B_1' & \text{i.e.} \quad F_2' = F_1'(B_2'/B_1') & 2\alpha_3 = A_1^* A_2 + A_2^* A_1 \\ F_2' B_3' = F_3' B_2' & \qquad F_3' = F_2'(B_3'/B_2') = F_1'(B_2'/B_1')(B_3'/B_2') & 2\alpha_1 = A_2^* A_3 + A_3^* A_2 \\ F_3' B_1' \neq F_1' B_3' & \text{i.e.} \quad F_3' \text{ is already fixed by relations above.} & 2\alpha_2 = A_3^* A_1 + A_1^* A_3 \end{array} \qquad (62)$$

Notice that we cannot set all three relations on the left side in this table; the first two determine the third. This means we cannot reduce the gh term to a left hand quaternion, the way we did in solving the eqn (2) above.

$$\begin{aligned} g &= A_1^* F_1' + A_2^* F_1'(B_2' B_1'^*)/|B_1|^2 + A_3^* F_1'(B_2' B_1'^*)(B_3' B_2'^*)/(|B_1|^2|B_2|^2) && (63) \\ gh &= |A_1|^2 F_1' B_1' + |A_2|^2 F_1'(B_2' B_1'^*) B_2'/|B_1|^2 + |A_3|^2 F_1'(B_2' B_1'^*)(B_3' B_2'^*) B_3'/(|B_1|^2|B_2|^2) && (64) \\ &+ 2\alpha_3 F_1' B_2' \\ &+ 2\alpha_1 F_1'(B_2' B_1'^*) B_3'/|B_1|^2 \\ &+ A_3^* A_1 F_1'(B_2' B_1'^*)(B_3' B_2'^*) B_1'/(|B_1|^2|B_2|^2) + (2\alpha_2 - A_3^* A_1) F_1' B_3' \end{aligned}$$

We can, however, reduce the gh term significantly, so that it consists of essentially the sum of two terms, one left hand quaternion plus one product of right and left handed factors. Let's introduce, G_1, G_2, so that,

$$\begin{aligned} gh &= G_1' + A_3^* A_1 G_2' && (65) \\ G_1' &= |A_1|^2 F_1' B_1' + |A_2|^2 F_1'(B_2' B_1'^*) B_2'/|B_1|^2 + |A_3|^2 F_1'(B_2' B_1'^*)(B_3' B_2'^*) B_3'/(|B_1|^2|B_2|^2) && (66) \\ &+ 2\alpha_3 F_1' B_2' + 2\alpha_1 F_1'(B_2' B_1'^*) B_3'/|B_1|^2 + 2\alpha_2 F_1' B_3' \\ G_2' &= F_1'(B_2' B_1'^*)(B_3' B_2'^*) B_1'/(|B_1|^2|B_2|^2) - F_1' B_3' && (67) \end{aligned}$$

then, $G_1' \in \mathbb{H}_L$, and, $G_2' \in \mathbb{H}_L$. So, even though we can't write, $(gh)^{-1} = (gh)^*/|gh|^2$, here, because $gh \in X_n$, we already have the inverse formula for this particular hexpe number in the previous solution given in eqn (40). Thus, comparing terms, we can immediately write down the required inverse, $(gh)^{-1} = (G_1' + A_3^* A_1 G_2')^{-1}$, we get,

$$(G_1' + A_3^* A_1 G_2')^{-1} = \left(\frac{\begin{array}{c} (|G_1|^2 + |A_3^* A_1|^2 |G_2|^2)(G_1'^* + (A_3^* A_1)^* G_2'^*) + |A_3^* A_1|^2 (G_2^* G_1 G_2^*)' \\ +(A_3^* A_1)^* (G_1^* G_2 G_1^*)' + |G_2|^2 (A_3^* A_1)^* (A_3^* A_1)^* G_1'^* + |G_1|^2 (A_3^* A_1) G_2'^* \end{array}}{\begin{array}{c} (|G_1|^2 - |A_3^* A_1|^2 |G_2|^2)^2 + \\ 4(\ |G_1|^2 + |A_3^* A_1|^2 |G_2|^2\ )\alpha\beta + 4|G_1|^2|G_2|^2\alpha^2 + 4|A_3^* A_1|^2 \beta^2 \end{array}} \right) \qquad (68)$$

$$\text{where, } 2\alpha = A_3^* A_1 + (A_3^* A_1)^*\ , \ \ 2\beta = G_1^* G_2 + (G_1^* G_2)^*$$

from which the inverse, $h^{-1} = (gh)^{-1} g$, follows.

Again, we pick $F_1' = +|B_1|$, to help simplify these expressions. Then,

$$g = A_1^*|B_1| + A_2^*(B_2'B_1'^*)/|B_1| + A_3^*(B_2'B_1'^*)(B_3'B_2'^*)/(|B_1||B_2|^2) \tag{69}$$

$$\begin{aligned} G_1' &= |A_1|^2|B_1|B_1' + |A_2|^2(B_2'B_1'^*)B_2'/|B_1| + |A_3|^2(B_2'B_1'^*)(B_3'B_2'^*)B_3'/(|B_1||B_2|^2) \\ &+ 2\alpha_3|B_1|B_2' + 2\alpha_1(B_2'B_1'^*)B_3'/|B_1| + 2\alpha_2|B_1|B_3' \end{aligned} \tag{70}$$

$$G_2' = (B_2'B_1'^*)(B_3'B_2'^*)B_1'/(|B_1||B_2|^2) - |B_1|B_3' \tag{71}$$

$$\begin{aligned} \therefore \quad |G_1|^2 &= G_1^*G_1 = G_1G_1^* \\ &= |A_1|^4|B_1|^4 + |A_2|^4|B_2|^4 + |A_3|^4|B_3|^4 \\ &- 2|A_1|^2|A_2|^2|B_1|^2|B_2|^2 - 2|A_2|^2|A_3|^2|B_2|^2|B_3|^2 \\ &+ |A_3|^2|A_1|^2(8\beta_1\beta_2\beta_3 - 4\beta_1^2|B_1|^2 - 4\beta_3^2|B_3|^2 + 2|B_1|^2|B_2|^2|B_3|^2)/|B_2|^2 \\ &+ 4\beta_1^2|A_2|^2|A_3|^2 + 4\beta_3^2|A_1|^2|A_2|^2 \\ &+ 4\alpha_1^2|B_2|^2|B_3|^2 + 4\alpha_2^2|B_1|^2|B_3|^2 + 4\alpha_3^2|B_1|^2|B_2|^2 \\ &+ 4|A_1|^2|B_1|^2(-\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) \\ &+ 4|A_2|^2|B_2|^2(+\alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3) \\ &+ 4|A_3|^2|B_3|^2(+\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3) \\ &+ 8\beta_2\beta_3\alpha_1|A_1|^2 + 8\beta_1\beta_3\alpha_2|A_2|^2 + 8\beta_1\beta_2\alpha_3|A_3|^2 \\ &+ 8\alpha_2\alpha_3\beta_1|B_1|^2 + 8\alpha_3\alpha_1\beta_2|B_2|^2 + 8\alpha_1\alpha_2\beta_3|B_3|^2 \end{aligned} \tag{72}$$

$$|G_2|^2 = G_2^*G_2 = (8\beta_1\beta_2\beta_3 - 4\beta_1^2|B_1|^2 - 4\beta_2^2|B_2|^2 - 4\beta_3^2|B_3|^2 + 4|B_1|^2|B_2|^2|B_3|^2)/|B_2|^2 \tag{73}$$

$$G_1^*G_2 + (G_1^*G_2)^* = -2\alpha_2(8\beta_1\beta_2\beta_3 - 4\beta_1^2|B_1|^2 - 4\beta_2^2|B_2|^2 - 4\beta_3^2|B_3|^2 + 4|B_1|^2|B_2|^2|B_3|^2)/|B_2|^2 = -2\alpha|G_2|^2 \tag{74}$$

where,

$$\begin{aligned} &2\alpha_3 = A_1^*A_2 + A_2^*A_1 && 2\beta_3 = B_1^*B_2 + B_2^*B_1 = B_1B_2^* + B_2B_1^* \\ &2\alpha_1 = A_2^*A_3 + A_3^*A_2 && 2\beta_1 = B_2^*B_3 + B_3^*B_2 = B_2B_3^* + B_3B_2^* \\ &2\alpha_2 = A_3^*A_1 + A_1^*A_3 = 2\alpha && 2\beta_2 = B_3^*B_1 + B_1^*B_3 = B_3B_1^* + B_1B_3^* \end{aligned} \tag{75}$$

Computing $G_1G_1^*$ manually is somewhat faster than computing $G_1^*G_1$. Our choice of index assignments on $\{\ \alpha_k, \beta_k\ \}$ follows the familiar cross product index cycling, so that they are easy to remember. However, when we extend this linear problem to arbitrary n in (4) the cross product index pattern is no longer useful; and, we will require new notation. When quaternion terms are paired up into scalar results, it is helpful to remember the cases where the scalar is invariant under cyclic permutations, i.e. $S(P^*Q\cdots RS^*T^*U) = S(UP^*Q\cdots RS^*T^*) = S(T^*UP^*Q\cdots RS^*)$, etc..also, the scalar is invariant under conjugation, i.e. $S(P^*Q\cdots RS^*T^*U) = S(U^*TSR^*\cdots Q^*P)$, etc...and again invariant under hand transformation, i.e. $S(P^*Q\cdots RS^*T^*U) = S(U'T'^*S'^*R'\cdots Q'P'^*)$, etc..., conjugation and hand transformation reversing the order of all the factors in the whole product expression. Substituting the (74) result for 2β in (68) simplifies the denominator, and, rearranging the numerator also, we can now re-write this formula;

$$(G_1' + A_3^*A_1G_2')^{-1} = \frac{\begin{pmatrix} (|G_1|^2 + 2\alpha(A_3^*A_1)^*|G_2|^2)\boldsymbol{G}_1'^* + (2\alpha|G_1|^2 + |A_3^*A_1|^2(A_3^*A_1)^*|G_2|^2)\boldsymbol{G}_2'^* \\ +|A_3^*A_1|^2(\boldsymbol{G}_2^*\boldsymbol{G}_1\boldsymbol{G}_2^*)' + (A_3^*A_1)^*(\boldsymbol{G}_1^*\boldsymbol{G}_2\boldsymbol{G}_1^*)' \end{pmatrix}}{(|G_1|^2 - |A_3^*A_1|^2|G_2|^2)^2} \tag{76}$$

where, $2\alpha = A_3^*A_1 + (A_3^*A_1)^*$, $\qquad \mathbb{H}_L$ in **bold**

Note that if $B_1B_3^*$ commutes with the product $B_2B_1^*$, then $|G_2|^2$ vanishes[7], etc.., and, $G_2 \equiv 0$, in which case, the inverse formula becomes the usual, $G_1'^{-1} = G_1'^*/|G_1|^2$, of one hand quaternions. Otherwise, when such pairs do not so commute, we have to use the formula (76) for the inverse. We can further reduce this formula, by using (74) again;

$$\begin{aligned} G_1^* G_2 + G_2^* G_1 &= -2\alpha |G_2|^2 \\ \Longrightarrow \quad G_2^* G_1 &= -2\alpha |G_2|^2 - G_1^* G_2 \quad \therefore \quad G_2^* G_1 G_2^* = -2\alpha |G_2|^2 G_2^* - |G_2|^2 G_1^* \\ \Longrightarrow \quad G_1^* G_2 &= -2\alpha |G_2|^2 - G_2^* G_1 \quad \therefore \quad G_1^* G_2 G_1^* = -2\alpha |G_2|^2 G_1^* - |G_1|^2 G_2^* \end{aligned} \tag{77}$$

Putting these results for the three factor products, $G_{\cdot}^* G_{\cdot} G_{\cdot}^*$, into formula (76), and simplifying, we get,

$$(G_1' + A_3^* A_1 G_2')^{-1} = \frac{G_1'^* + A_3^* A_1 G_2'^*}{|G_1|^2 - |A_3^* A_1|^2 |G_2|^2} \tag{78}$$

In a previous paper[2] [PJ3] we introduced the concept of right conjugate and left conjugate, h^{*R} and h^{*L}, which attack the right hand quaternion component and left hand quaternion component of a two-hand quaternion, h, separately. Note, the normal conjugate, h^*, acts on both right and left hand components simultaneously. Using this notation,

$$(G_1' + A_3^* A_1 G_2')^{*L} (G_1' + A_3^* A_1 G_2') = (G_1'^* + A_3^* A_1 G_2'^*)(G_1' + A_3^* A_1 G_2') \tag{79}$$

$$= G_1'^* G_1' + A_3^* A_1 G_1'^* G_2' + A_3^* A_1 G_2'^* G_1' + (A_3^* A_1)^2 G_2'^* G_2' \tag{80}$$

$$= |G_1|^2 + A_3^* A_1 (G_2 G_1^* + G_1 G_2^*) + (A_3^* A_1)^2 |G_2|^2 \tag{81}$$

$$= |G_1|^2 + A_3^* A_1 (-2\alpha |G_2|^2) + (A_3^* A_1)^2 |G_2|^2 = |G_1|^2 - (2\alpha - (A_3^* A_1))(A_3^* A_1)|G_2|^2 \tag{82}$$

$$= |G_1|^2 - (A_3^* A_1)^* (A_3^* A_1) |G_2|^2 = |G_1|^2 - |A_3^* A_1|^2 |G_2|^2 \tag{83}$$

and we can write this inverse,

$$(G_1' + A_3^* A_1 G_2')^{-1} = \frac{(G_1' + A_3^* A_1 G_2')^{*L}}{|G_1|^2 - |A_3^* A_1|^2 |G_2|^2} = \frac{(G_1' + A_1^* A_3 G_2')^*}{|G_1|^2 - |A_3^* A_1|^2 |G_2|^2} \tag{84}$$

This construction relies on the fact that, $G_1^* G_2 + G_2^* G_1 = -2\alpha |G_2|^2$, found above. If $G_1 = 0$, then, $-2\alpha |G_2|^2 = 0$, so either, $2\alpha = 0$ or $|G_2|^2 = 0$. If $2\alpha = 0$, then $A_3^* A_1$ is a pure quaternion, i.e. a vector in Hamilton's calculus, $A_3^* A_1 = V(A_3^* A_1)$, and its square is negative, $(A_3^* A_1)^2 = -|A_3^* A_1|^2$, and its conjugate is obtained from a sign flip, $(A_3^* A_1)^* = -(A_3^* A_1)$; the inverse formula reduces to, $(A_3^* A_1 G_2')^{-1} = [(A_3^* A_1)^* / |A_3^* A_1|^2][G_2'^* / |G_2|^2]$, which we can write, $= (A_3^* A_1)^{-1} G_2'^{-1}$, which is what we'd expect. If both $G_1 = 0$ and $G_2 = 0$, then the denominator vanishes, but the L-H-S is zero also, i.e. there's nothing to invert. If $G_1 \neq 0$, and either $G_2 = 0$ or $A_3^* A_1 = 0$, then we just get the one hand inverse, $G_1'^{-1} = G_1'^* / |G_1|^2$. Finally, if all three factors, $G_1', G_2', A_3^* A_1$, are non-zero, but, $|G_1| = |A_3^* A_1 G_2|$, then the denominator vanishes, and there's no inverse for this two-hand quaternion. Apart from these special situations, the inverse exists, is a general two-hand quaternion, and given by the formulas (78) and (84).

In the final step, we need to construct, $h^{-1} = (G_1' + A_3^* A_1 G_2')^{-1} g = (gh)^{*L} g / \lambda$, using the definitions (69)-(73), where, the scalar, $\lambda = (gh)^{*L}(gh)$, is the denominator in (84). Substituting and rearranging yields $3^3 = 27$ terms for the numerator, with the form $A_{\cdot}^* A_{\cdot} A_{\cdot}^* B_{\cdot}'^* B_{\cdot}' B_{\cdot}'^*$, which we can write[8],

$$\begin{aligned} (gh)^{*L} g = \; & A_1^* A_1 A_1^* \; B_1'^* B_1' B_1'^* + A_2^* A_1 A_1^* \; B_1'^* B_2' B_1'^* + A_3^* A_1 A_1^* \; B_1'^* B_3' B_1'^* \\ + \; & A_1^* A_1 A_2^* \; B_1'^* B_1' B_2'^* + A_2^* A_1 A_2^* \; B_2'^* B_2' B_1'^* + A_3^* A_1 A_2^* \; B_1'^* B_2' B_3'^* \\ + \; & A_1^* A_1 A_3^* \; B_1'^* B_1' B_3'^* + A_2^* A_1 A_3^* \; B_1'^* B_3' B_2'^* + A_3^* A_1 A_3^* \; B_1'^* B_3' B_3'^* \\ + \; & A_1^* A_2 A_1^* \; B_1'^* B_1' B_2'^* + A_2^* A_2 A_1^* \; B_2'^* B_2' B_1'^* + A_3^* A_2 A_1^* \; B_3'^* B_1' B_2'^* \\ + \; & A_1^* A_2 A_2^* \; B_2'^* B_1' B_2'^* + A_2^* A_2 A_2^* \; B_2'^* B_2' B_2'^* + A_3^* A_2 A_2^* \; B_2'^* B_3' B_2'^* \\ + \; & A_1^* A_2 A_3^* \; B_1'^* B_3' B_2'^* + A_2^* A_2 A_3^* \; B_2'^* B_2' B_3'^* + A_3^* A_2 A_3^* \; B_3'^* B_3' B_2'^* \\ + \; & A_1^* A_3 A_1^* \; B_1'^* B_1' B_3'^* + A_2^* A_3 A_1^* \; B_3'^* B_1' B_2'^* + A_3^* A_3 A_1^* \; B_3'^* B_3' B_1'^* \\ + \; & A_1^* A_3 A_2^* \; B_3'^* B_2' B_1'^* + A_2^* A_3 A_2^* \; B_3'^* B_2' B_2'^* + A_3^* A_3 A_2^* \; B_3'^* B_3' B_2'^* \\ + \; & A_1^* A_3 A_3^* \; B_3'^* B_1' B_3'^* + A_2^* A_3 A_3^* \; B_3'^* B_2' B_3'^* + A_3^* A_3 A_3^* \; B_3'^* B_3' B_3'^* \end{aligned} \tag{85}$$

Once again, the solution for q has the form,

$$q = \frac{\sum A_{\cdot}^* A_{\cdot} A_{\cdot}^* C B_{\cdot}^* B_{\cdot} B_{\cdot}^*}{\lambda} \tag{86}$$

with, $\lambda = (gh)^{*L}(gh)$, this time, instead of the previous scalar, $|gh|^2 = (gh)^*(gh)$, in the denominator (31); but, since that previous $gh \in \mathbb{H}_L$, we could also have written that scalar there using the left conjugate, $|gh|^2 = (gh)^{*L}(gh)$.

In working out these formulas, we have to simplify many multi-factor B-products, requiring us to use various expression block reduction techniques, e.g.,

$$
\begin{aligned}
& B_1B_2^*B_1B_3^*B_2B_3^* + (B_1B_2^*B_1B_3^*B_2B_3^*)^* \\
+ \;& B_2^*B_1B_3^*B_2B_1^*B_3 + (B_2^*B_1B_3^*B_2B_1^*B_3)^* \\
\\
= \;& B_1B_2^*B_1B_3^*B_2B_3^* + (B_1B_2^*B_1B_3^*B_2B_3^*)^* \\
+ \;& B_3^*B_1B_2^*B_3B_1^*B_2 + (B_3^*B_1B_2^*B_3B_1^*B_2)^* \qquad \leftarrow \text{swap conjugate }^* \\
\\
= \;& B_1B_2^*B_1B_3^*B_2B_3^* + (B_1B_2^*B_1B_3^*B_2B_3^*)^* \\
+ \;& B_1B_2^*B_3B_1^*B_2B_3^* + (B_1B_2^*B_3B_1^*B_2B_3^*)^* \qquad \leftarrow \text{rotate } B\text{'s} \\
\\
= \;& B_1B_2^*2\beta_2B_2B_3^* + (B_1B_2^*2\beta_2B_2B_3^*)^* \qquad \leftarrow \text{add central pair} \\
= \;& 2\beta_2|B_2|^2(B_1B_3^* + (B_1B_3^*)^*) \\
= \;& (2\beta_2)^2|B_2|^2
\end{aligned}
$$

then we have,

$$
\begin{aligned}
\lambda \;=\; & |G_1|^2 - |A_3^*A_1|^2|G_2|^2 \qquad (87) \\
\\
+ \;& |A_1|^4|B_1|^4 + |A_2|^4|B_2|^4 + |A_3|^4|B_3|^4 - 2|A_1|^2|A_2|^2|B_1|^2|B_2|^2 - 2|A_2|^2|A_3|^2|B_2|^2|B_3|^2 - 2|A_3|^2|A_1|^2|B_3|^2|B_1|^2 \\
+ \;& 4\beta_1^2|A_2|^2|A_3|^2 + 4\beta_2^2|A_3|^2|A_1|^2 + 4\beta_3^2|A_1|^2|A_2|^2 + 4\alpha_1^2|B_2|^2|B_3|^2 + 4\alpha_2^2|B_3|^2|B_1|^2 + 4\alpha_3^2|B_1|^2|B_2|^2 \\
+ \;& 4|A_1|^2|B_1|^2(-\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) + 4|A_2|^2|B_2|^2(+\alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3) + 4|A_3|^2|B_3|^2(+\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3) \\
+ \;& 8\alpha_1\beta_2\beta_3|A_1|^2 + 8\alpha_2\beta_3\beta_1|A_2|^2 + 8\alpha_3\beta_1\beta_2|A_3|^2 + 8\alpha_2\alpha_3\beta_1|B_1|^2 + 8\alpha_3\alpha_1\beta_2|B_2|^2 + 8\alpha_1\alpha_2\beta_3|B_3|^2
\end{aligned}
$$

where, $\alpha_k,\ \beta_k,$ are defined in (75).

Next, the 27 terms in the numerator are reduced to 15, the last 6 terms being kept in the $A^*_{\cdot}\,A_{\cdot}\,A^*_{\cdot}\,B'^*_{\cdot}\,B'_{\cdot}\,B'^*_{\cdot}$ form.

$$
\begin{aligned}
(gh)^{*L}g \;=\; & (\,+|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2\,)A_1^*B_1'^* \\
+ \;& (\,-|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2\,)A_2^*B_2'^* \\
+ \;& (\,-|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 + |A_3|^2|B_3|^2\,)A_3^*B_3'^* \\
\\
+ \;& (|B_1|^2 2\alpha_3 + |A_2|^2 2\beta_3)A_1^*B_2'^* \;+\; (|B_2|^2 2\alpha_3 + |A_1|^2 2\beta_3)A_2^*B_1'^* \\
+ \;& (|B_2|^2 2\alpha_1 + |A_3|^2 2\beta_1)A_2^*B_3'^* \;+\; (|B_3|^2 2\alpha_1 + |A_2|^2 2\beta_1)A_3^*B_2'^* \\
+ \;& (|B_3|^2 2\alpha_2 + |A_1|^2 2\beta_2)A_3^*B_1'^* \;+\; (|B_1|^2 2\alpha_2 + |A_3|^2 2\beta_2)A_1^*B_3'^* \qquad (88) \\
\\
+ \;& A_1^*\,A_2\,A_3^*\,B_1'^*\,B_3'\,B_2'^* \\
+ \;& A_1^*\,A_3\,A_2^*\,B_3'^*\,B_2'\,B_1'^* \\
+ \;& A_2^*\,A_1\,A_3^*\,B_1'^*\,B_3'\,B_2'^* \\
+ \;& A_2^*\,A_3\,A_1^*\,B_3'^*\,B_1'\,B_2'^* \\
+ \;& A_3^*\,A_1\,A_2^*\,B_1'^*\,B_2'\,B_3'^* \\
+ \;& A_3^*\,A_2\,A_1^*\,B_3'^*\,B_1'\,B_2'^*
\end{aligned}
$$

These last six can also be written as 4, further reducing the number of numerator terms to 13,

$$
A_3^*\,A_1\,A_2^*\,B_1'^*\,B_2'\,B_3'^* + 2\alpha_3\,A_3^*\,B_1'^*\,B_3'\,B_2'^* + 2\alpha_1\,A_1^*\,B_3'^*\,B_1'\,B_2'^* + A_1^*\,A_3\,A_2^*\,B_3'^*\,B_2'\,B_1'^* \qquad (88a)
$$

These formulas, (87) and (88), are invariant under index exchange. Most of the expressions are obviously unchanged when we swap any pair of indicies, but the last six terms in (88) require some calculation to demonstrate this fact. Each of the three blocks of expressions—separated by blank lines—in eqn (88), is independently invariant under index exchange. For example, to swap the $1-2$ indicies, we make exchanges: $A_1 \to A_2,\ A_2 \to A_1$ and $B_1 \to B_2,\ B_2 \to B_1$, which causes the scalar exchanges, $\alpha_1 \to \alpha_2,\ \alpha_2 \to \alpha_1,\ \alpha_3 \to \alpha_3$ and $\beta_1 \to \beta_2,\ \beta_2 \to \beta_1,\ \beta_3 \to \beta_3$, whence a simple inspection verifies that the first two expression blocks are unchanged after this swap.

With the last block, the easiest way to demonstrate invariance is to swap indicies and subtract the original block, and then show that the difference can be reduced to zero. Let's do this with the equivalent four term expression.

Swap Indicies: 1-2

$$\begin{aligned}
&(A_3^* A_2 A_1^* B_2'^* B_1' B_3'^* + 2\alpha_3 A_3^* B_2'^* B_3' B_1'^* + 2\alpha_2 A_2^* B_3'^* B_2' B_1'^* + A_2^* A_3 A_1^* B_3'^* B_1' B_2'^*) \\
-&(A_3^* A_1 A_2^* B_1'^* B_2' B_3'^* + 2\alpha_3 A_3^* B_1'^* B_3' B_2'^* + 2\alpha_1 A_1^* B_3'^* B_1' B_2'^* + A_1^* A_3 A_2^* B_3'^* B_2' B_1'^*)
\end{aligned}$$

$$\Longrightarrow \quad A_2^* A_3 A_1^* B_3'^* B_1' B_2'^* = (2\alpha_1 - A_3^* A_2) A_1^* B_3'^* B_1' B_2'^* = 2\alpha_1 A_1^* B_3'^* B_1' B_2'^* - A_3^* A_2 A_1^* B_3'^* B_1' B_2'^*$$

$$\Longrightarrow \quad A_1^* A_3 A_2^* B_3'^* B_2' B_1'^* = (2\alpha_2 - A_3^* A_1) A_2^* B_3'^* B_2' B_1'^* = 2\alpha_2 A_2^* B_3'^* B_2' B_1'^* - A_3^* A_1 A_2^* B_3'^* B_2' B_1'^*$$

$$\begin{aligned}
=&(A_3^* A_2 A_1^* B_2'^* B_1' B_3'^* + 2\alpha_3 A_3^* B_2'^* B_3' B_1'^* + \quad 0 \quad - A_3^* A_2 A_1^* B_3'^* B_1' B_2'^*) \\
-&(A_3^* A_1 A_2^* B_1'^* B_2' B_3'^* + 2\alpha_3 A_3^* B_1'^* B_3' B_2'^* + \quad 0 \quad - A_3^* A_1 A_2^* B_3'^* B_2' B_1'^*)
\end{aligned}$$

$$\begin{aligned}
=&(A_3^* A_2 A_1^* (B_2'^* B_1' B_3'^* - B_3'^* B_1' B_2'^*) + 2\alpha_3 A_3^* B_2'^* B_3' B_1'^*) \\
-&(A_3^* A_1 A_2^* (B_1'^* B_2' B_3'^* - B_3'^* B_2' B_1'^*) + 2\alpha_3 A_3^* B_1'^* B_3' B_2'^*)
\end{aligned}$$

$$\Longrightarrow \quad B_2'^* B_1' B_3'^* - B_3'^* B_1' B_2'^* = B_2'^* (2\beta_2 - B_3' B_1'^*) - (2\beta_2 - B_1'^* B_3') B_2'^* = B_1'^* B_3' B_2'^* - B_2'^* B_3' B_1'^*$$

$$\Longrightarrow \quad B_1'^* B_2' B_3'^* - B_3'^* B_2' B_1'^* = B_1'^* (2\beta_1 - B_3' B_2'^*) - (2\beta_1 - B_2'^* B_3') B_1'^* = B_2'^* B_3' B_1'^* - B_1'^* B_3' B_2'^*$$

$$\begin{aligned}
=&(A_3^* A_2 A_1^* B_1'^* B_3' B_2'^* - A_3^* A_2 A_1^* B_2'^* B_3' B_1'^* + 2\alpha_3 A_3^* B_2'^* B_3' B_1'^*) \\
-&(A_3^* A_1 A_2^* B_2'^* B_3' B_1'^* - A_3^* A_1 A_2^* B_1'^* B_3' B_2'^* + 2\alpha_3 A_3^* B_1'^* B_3' B_2'^*)
\end{aligned}$$

$$\Longrightarrow \quad A_3^* A_2 A_1^* B_1'^* B_3' B_2'^* + A_3^* A_1 A_2^* B_1'^* B_3' B_2'^* = (A_3^* A_2 A_1^* + A_3^* A_1 A_2^*) B_1'^* B_3' B_2'^* = A_3^* 2\alpha_3 B_1'^* B_3' B_2'^*$$

$$\Longrightarrow \quad A_3^* A_2 A_1^* B_2'^* B_3' B_1'^* + A_3^* A_1 A_2^* B_2'^* B_3' B_1'^* = (A_3^* A_2 A_1^* + A_3^* A_1 A_2^*) B_2'^* B_3' B_1'^* = A_3^* 2\alpha_3 B_2'^* B_3' B_1'^*$$

$$\begin{aligned}
=&(A_3^* 2\alpha_3 B_1'^* B_3' B_2'^* + 2\alpha_3 A_3^* B_2'^* B_3' B_1'^*) \\
-&(A_3^* 2\alpha_3 B_2'^* B_3' B_1'^* + 2\alpha_3 A_3^* B_1'^* B_3' B_2'^*)
\end{aligned}$$

$$= \ 0 \qquad\qquad \text{Q.E.D.}$$

Similarly, the block can be shown to remain unchanged under 1-3 and 2-3 index swaps. We can express this block in yet another way, where the the number of six factor quaternion terms, $A^*_. A_. A^*_. B'^*_. B'_. B'^*_.$, becomes just one, but we need to introduce more terms of lower order to accomplish this; an example is shown below:

$$\begin{aligned}
&+ 2A_1^* A_2 A_3^* B_1'^* B_2' B_3'^* \\
&+ (-2\alpha_1 A_1^* + 2\alpha_2 A_2^* - 2\alpha_3 A_3^*) B_1'^* B_2' B_3'^* + A_1^* A_2 A_3^* (-2\beta_1 B_1'^* + 2\beta_2 B_2'^* - 2\beta_3 B_3'^*) \\
&+ 2\alpha_3 2\beta_1 A_3^* B_1'^* + 2\alpha_1 2\beta_3 A_1^* B_3'^*
\end{aligned} \tag{88b}$$

The last 2 of these 9 terms can be absorbed into the middle expression block of eqn (88), but that still leaves 7 terms, compared to the 6 $A^*_. A_. A^*_. B'^*_. B'_. B'^*_.$, orignal terms, so we don't get a reduction of term count. We do get a reduction in the "order" of each of five terms, and sometimes this can be useful. However, we have chosen to keep these six original bi-cubic quaternion terms in their initial form.

Hence, both the numerator and denominator in $h^{-1} = (gh)^{*L} g/\lambda$, are separately invariant under the exchange of any two original indicies: 1-2, 2-3, or 3-1. This is expected because the $A_. q B_.$ terms in eqn (3) can be arranged in any order owing to the associativity of the addition operator in quaternion algebra. We are now ready to write down the solution to the "three term" linear problem.

Hence, for the " THREE TERM " linear problem,

$$A_1 q B_1 + A_2 q B_2 + A_3 q B_3 \; = \; C \tag{3}$$

the two-hand inverse factor, h^{-1}, is,

$$(A_1 B'_1 + A_2 B'_2 + A_3 B'_3)^{-1} \;\; = \;\; \left(\frac{\begin{array}{l} (\, +|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 \,) A_1^* B_1'^* \quad + \\ (\, -|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 \,) A_2^* B_2'^* \quad + \\ (\, -|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 + |A_3|^2|B_3|^2 \,) A_3^* B_3'^* \\ +(|B_1|^2 2\alpha_3 + |A_2|^2 2\beta_3) A_1^* B_2'^* \;\; + \;\; (|B_2|^2 2\alpha_3 + |A_1|^2 2\beta_3) A_2^* B_1'^* \\ +(|B_2|^2 2\alpha_1 + |A_3|^2 2\beta_1) A_2^* B_3'^* \;\; + \;\; (|B_3|^2 2\alpha_1 + |A_2|^2 2\beta_1) A_3^* B_2'^* \\ +(|B_3|^2 2\alpha_2 + |A_1|^2 2\beta_2) A_3^* B_1'^* \;\; + \;\; (|B_1|^2 2\alpha_2 + |A_3|^2 2\beta_2) A_1^* B_3'^* \\ +A_1^* \, A_2 \, A_3^* \, B_1'^* \, B_3' \, B_2'^* + A_1^* \, A_3 \, A_2^* \, B_3'^* \, B_2' \, B_1'^* + A_2^* \, A_1 \, A_3^* \, B_1'^* \, B_3' \, B_2'^* \\ +A_2^* \, A_3 \, A_1^* \, B_3'^* \, B_1' \, B_2'^* + A_3^* \, A_1 \, A_2^* \, B_1'^* \, B_2' \, B_3'^* + A_3^* \, A_2 \, A_1^* \, B_3'^* \, B_1' \, B_2'^* \end{array}}{\begin{array}{l} |A_1|^4|B_1|^4 + |A_2|^4|B_2|^4 + |A_3|^4|B_3|^4 \\ -2|A_1|^2|A_2|^2|B_1|^2|B_2|^2 - 2|A_2|^2|A_3|^2|B_2|^2|B_3|^2 - 2|A_3|^2|A_1|^2|B_3|^2|B_1|^2 \\ +4\beta_1^2|A_2|^2|A_3|^2 + 4\beta_2^2|A_3|^2|A_1|^2 + 4\beta_3^2|A_1|^2|A_2|^2 \\ +4\alpha_1^2|B_2|^2|B_3|^2 + 4\alpha_2^2|B_3|^2|B_1|^2 + 4\alpha_3^2|B_1|^2|B_2|^2 \\ +4|A_1|^2|B_1|^2(-\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) \\ +4|A_2|^2|B_2|^2(\;\;\alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3) \\ +4|A_3|^2|B_3|^2(\;\;\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3) \\ +8\beta_2\beta_3\alpha_1|A_1|^2 + 8\beta_3\beta_1\alpha_2|A_2|^2 + 8\beta_1\beta_2\alpha_3|A_3|^2 \\ +8\alpha_2\alpha_3\beta_1|B_1|^2 + 8\alpha_3\alpha_1\beta_2|B_2|^2 + 8\alpha_1\alpha_2\beta_3|B_3|^2 \end{array}} \right) \tag{89}$$

and the solution is,

$$q \;\; = \;\; \left(\frac{\begin{array}{l} (\, +|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 \,) A_1^* C B_1^* \quad + \\ (\, -|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 \,) A_2^* C B_2^* \quad + \\ (\, -|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 + |A_3|^2|B_3|^2 \,) A_3^* C B_3^* \\ +(|B_1|^2 2\alpha_3 + |A_2|^2 2\beta_3) A_1^* C B_2^* \;\; + \;\; (|B_2|^2 2\alpha_3 + |A_1|^2 2\beta_3) A_2^* C B_1^* \\ +(|B_2|^2 2\alpha_1 + |A_3|^2 2\beta_1) A_2^* C B_3^* \;\; + \;\; (|B_3|^2 2\alpha_1 + |A_2|^2 2\beta_1) A_3^* C B_2^* \\ +(|B_3|^2 2\alpha_2 + |A_1|^2 2\beta_2) A_3^* C B_1^* \;\; + \;\; (|B_1|^2 2\alpha_2 + |A_3|^2 2\beta_2) A_1^* C B_3^* \\ +A_1^* \, A_2 \, A_3^* \, C \, B_2^* \, B_3 \, B_1^* + A_1^* \, A_3 \, A_2^* \, C \, B_1^* \, B_2 \, B_3^* + A_2^* \, A_1 \, A_3^* \, C \, B_2^* \, B_3 \, B_1^* \\ +A_2^* \, A_3 \, A_1^* \, C \, B_2^* \, B_1 \, B_3^* + A_3^* \, A_1 \, A_2^* \, C \, B_3^* \, B_2 \, B_1^* + A_3^* \, A_2 \, A_1^* \, C \, B_2^* \, B_1 \, B_3^* \end{array}}{\begin{array}{l} |A_1|^4|B_1|^4 + |A_2|^4|B_2|^4 + |A_3|^4|B_3|^4 \\ -2|A_1|^2|A_2|^2|B_1|^2|B_2|^2 - 2|A_2|^2|A_3|^2|B_2|^2|B_3|^2 - 2|A_3|^2|A_1|^2|B_3|^2|B_1|^2 \\ +4\beta_1^2|A_2|^2|A_3|^2 + 4\beta_2^2|A_3|^2|A_1|^2 + 4\beta_3^2|A_1|^2|A_2|^2 \\ +4\alpha_1^2|B_2|^2|B_3|^2 + 4\alpha_2^2|B_3|^2|B_1|^2 + 4\alpha_3^2|B_1|^2|B_2|^2 \\ +4|A_1|^2|B_1|^2(-\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) \\ +4|A_2|^2|B_2|^2(\;\;\alpha_1\beta_1 - \alpha_2\beta_2 + \alpha_3\beta_3) \\ +4|A_3|^2|B_3|^2(\;\;\alpha_1\beta_1 + \alpha_2\beta_2 - \alpha_3\beta_3) \\ +8\beta_2\beta_3\alpha_1|A_1|^2 + 8\beta_3\beta_1\alpha_2|A_2|^2 + 8\beta_1\beta_2\alpha_3|A_3|^2 \\ +8\alpha_2\alpha_3\beta_1|B_1|^2 + 8\alpha_3\alpha_1\beta_2|B_2|^2 + 8\alpha_1\alpha_2\beta_3|B_3|^2 \\ +8\alpha_3\alpha_1\beta_2|B_1|^2 + 8\alpha_1\alpha_2\beta_3|B_2|^2 + 8\alpha_2\alpha_3\beta_1|B_3|^2 \end{array}} \right) \tag{90}$$

where,

$$\begin{array}{lll} 2\alpha_1 = A_2^* A_3 + A_3^* A_2 & 2\alpha_2 = A_3^* A_1 + A_1^* A_3 & 2\alpha_3 = A_1^* A_2 + A_2^* A_1 \\ 2\beta_1 = B_2^* B_3 + B_3^* B_2 & 2\beta_2 = B_3^* B_1 + B_1^* B_3 & 2\beta_3 = B_1^* B_2 + B_2^* B_1 \end{array}$$

$$A_k, B_k, C, q \in \mathbb{H}_R ; \;\; B'_k \in \mathbb{H}_L ; \;\; \alpha_k, \beta_k, \in \mathbb{R}; \quad k = 1, 2, 3.$$

Consider equation (4) with $n=4$,

$$\begin{aligned} A_1qB_1 + A_2qB_2 + \cdots + A_nqB_n &= C \qquad (4)\\ A_1B_1'\hat{q} + A_2B_2'\hat{q} + A_3B_3'\hat{q} + A_4B_4'\hat{q} &= \hat{C}\\ (A_1B_1' + A_2B_2' + A_3B_3' + A_4B_4')\hat{q} &= \hat{C}\\ \hat{q} &= (A_1B_1' + A_2B_2' + A_3B_3' + A_4B_4')^{-1}\hat{C} \end{aligned}$$

Proceeding as usual, with new parameters, $h, g, F_k; k = 1,2,3,4$; we have,

$$\begin{aligned} h &= A_1B_1' + A_2B_2' + A_3B_3' + A_4B_4' \qquad (91)\\ g &= A_1^*F_1' + A_2^*F_2' + A_3^*F_3' + A_4^*F_4' \qquad (92)\\ \therefore \quad gh &= |A_1|^2F_1'B_1' + |A_2|^2F_2'B_2' + |A_3|^2F_3'B_3' + |A_4|^2F_4'B_4' \qquad (93)\\ &+ A_1^*A_2F_1'B_2' + A_2^*A_1F_2'B_1'\\ &+ A_1^*A_3F_1'B_3' + A_3^*A_1F_3'B_1'\\ &+ A_1^*A_4F_1'B_4' + A_4^*A_1F_4'B_1'\\ &+ A_2^*A_3F_2'B_3' + A_3^*A_2F_3'B_2'\\ &+ A_2^*A_4F_2'B_4' + A_4^*A_2F_4'B_2'\\ &+ A_3^*A_4F_3'B_4' + A_4^*A_3F_4'B_3' \end{aligned}$$

Let's pick, F_1', F_2', F_3', F_4', and introduce the scalars, $\alpha_{jk} = A_j^*A_k + A_k^*A_j$, where,

$$\begin{array}{lll} F_1'B_2' = F_2'B_1' & \text{i.e. } F_2' = F_1'(B_2'/B_1') & 2\alpha_{12} = A_1^*A_2 + A_2^*A_1\\ F_1'B_3' = F_3'B_1' & \phantom{\text{i.e. }} F_3' = F_1'(B_3'/B_1') & 2\alpha_{13} = A_1^*A_3 + A_3^*A_1\\ F_1'B_4' = F_4'B_1' & \phantom{\text{i.e. }} F_4' = F_1'(B_4'/B_1') & 2\alpha_{14} = A_1^*A_4 + A_4^*A_1\\ F_2'B_3' \neq F_3'B_2' & \text{i.e. } F_3' \text{ is already fixed by relations above.} & 2\alpha_{23} = A_2^*A_3 + A_3^*A_2\\ F_2'B_4' \neq F_4'B_2' & \text{i.e. } F_4' \text{ is already fixed by relations above.} & 2\alpha_{24} = A_2^*A_4 + A_4^*A_2\\ F_3'B_4' \neq F_4'B_3' & \text{i.e. } F_4' \text{ is already fixed by relations above.} & 2\alpha_{34} = A_3^*A_4 + A_4^*A_3 \end{array} \qquad (94)$$

We now change our notation for the usual scalars from single index, α_j, to double index, α_{jk}, because the cross product index cycling permutations are no longer useful here. Things are very different this time around. We only have four free parameters, F_k, to set, but six relations that require them, to reduce the gh expression. The best we can do is to set, G_1, G_2, G_3, G_4, so that,

$$gh = G_1' + A_2^*A_3G_2' + A_2^*A_4G_3' + A_3^*A_4G_4' \qquad (95)$$

$$\begin{aligned} G_1' &= +|A_1|^2F_1'B_1' + |A_2|^2F_1'(B_2'/B_1')B_2' + |A_3|^2F_1'(B_3'/B_1')B_3' + |A_4|^2F_1'(B_4'/B_1')B_4' \qquad (96)\\ &+ 2\alpha_{12}F_1'B_2'\\ &+ 2\alpha_{13}F_1'B_3'\\ &+ 2\alpha_{14}F_1'B_4'\\ &+ 2\alpha_{23}F_1'(B_3'/B_1')B_2' + 2\alpha_{24}F_1'(B_4'/B_1')B_2' + 2\alpha_{34}F_1'(B_4'/B_1')B_3' \end{aligned}$$

$$\begin{aligned} G_2' &= F_1'((B_2'/B_1')B_3' - (B_3'/B_1')B_2') \qquad (97)\\ G_3' &= F_1'((B_2'/B_1')B_4' - (B_4'/B_1')B_2') \qquad (98)\\ G_4' &= F_1'((B_3'/B_1')B_4' - (B_4'/B_1')B_3') \qquad (99) \end{aligned}$$

where, $G_1', G_2', G_3', G_4' \in \mathbb{H}_L$. So, by introducing the four term g, we haven't achieved any reduction of the original h factor at all. We are right back to a four term problem (95), which is where we started out in (91). We could just as well simply multiply eqn (91) by A_1^{-1} and we'd obtain precisely the form of gh, with much less work. Now, there's nothing that says the g in (92) has to have four terms. It could have more. Also, the form of each term could be modified to be more effective in reducing the gh expression. We could also try different $g's$, say, g_1 and g_2, and add them to arrive at the required reduced expression, and so on. But, guessing is tedious, and we need a more structured approach. Beyond $n > 3$, therefore, our current method of approach is too complicated, even though it appeared simple at the first. We'll introduce a simpler and more direct method of obtaining the inverse h^{-1} in the sections below, where all these suggestive ideas become incorporated into the general method that solves the linear problem for all n. These first few solutions then form a useful check and guide to the development of the next method.

In our above examples, the strategy was to multiply the original bilinear form, $h = (A_1 B_1' + A_2 B_2' + \cdots + A_n B_n')$, by factors containing conjugates of the right hand quaternions, $g = (A_1^* F_1' + A_2^* F_2' + \cdots + A_n^* F_n')$, so that we could reduce the right hand quaternion components to scalars, using the two rules, $A_s^* A_s = |A_s|^2 \in \mathbb{R}$, when the indicies are the same, and, $A_s^* A_r + A_r^* A_s \in \mathbb{R}$, when the indicies differ. When we're completely successful, the product gh is a left handed quaternion, and we can immediately invert the linear equation, using the familiar one-handed quaternion algebra. If we're only partially successful, then we do not quite get a left handed quaternion, there's some right handed components remaining, but we can still solve the problem, because we can use the solution of the previous n-1 term problem to solve the n term one. This is the general idea. But, at the 4 term problem, things get too complex. There's no reduction at all, and we have to start being more imaginative about our choices of those free parameter factors. Now, the idea of constructing factors with conjugates of the right hand quaternions, could be represented more efficiently, notationally, by the equivalent idea of partial conjugation. We could just as well select, $g = (A_1 F_1' + A_2 F_2' + \cdots + A_n F_n')$, and then write, $g^{*R} = (A_1^* F_1' + A_2^* F_2' + \cdots + A_n^* F_n')$. We noticed that the solution to the "three term" problem could be effectively written using the left conjugate, $(gh)^{*L} g$, in the numerator, and, $\lambda = (gh)^{*L}(gh)$, in the denominator, as in eqn (86). So, the idea of the partial conjugate is a useful shorthand for the concept of conjugating just the right, or just the left, handed parameters. This notational abbreviation allows us to avoid writing things out explicitly all the time; and becomes the central convenience technique in the method we'll develop next, enabling us to manipulate the formulas much easier than would otherwise be the case. But, first, we need some additional notation also, to facilitate formula constructions in the later sections to follow.

DOUBLE DOT h: Let us define $\ddot{h}_n^{-1} \equiv \ddot{h}(A,B)_n^{-1}$ to be the explicit solution to the n-term linear problem, i.e.,

$$\ddot{h}_n = (A_1 B_1' + A_2 B_2' + \cdots + A_n B_n') \tag{100}$$

$$\ddot{h}_n^{-1} = (A_1 B_1' + A_2 B_2' + \cdots + A_n B_n')^{-1} \tag{101}$$

We put a double dot $\ddot{}$ above the h to distinguish this subscripted $\ddot{h}_n$ from the components of h, since, for the latter we've also previously used various subscripted variations like, $h_j, j = 1, 2, ..., 16$ and $h_0, h_{R1}, h_{R2}, ..., h_{Z2}, h_{Z3}$, and so on. The double dot $\ddot{}$ then indicates the double handed "bilinear" form of the hexpe number with n terms. Except for the appendix, we don't make reference to the components of an hexpe number in this paper. But, to avoid confusion with past papers, and any future works, we need to make this distinction. The double dot is included to *emphasize* the double handed bilinear format, and can appear on any such variable, $\ddot{h}$ or $\ddot{h}_n$, but is only needed on the subscripted varieties to disambiguate the multiple use of subscripts in these cases. The context usually determines whether h is in bilinear form or otherwise in basis component format. It should be noted that the latter format is, in fact, just a special bilinear form with 16 terms, $\ddot{h}_{16}$, where the handed quaternions involved are trivially proportional to the unit quaternions. But, when we use the double dot form, $\ddot{h}_{16}$, we're specifically emphasizing that the quaternions in the 16 terms are not necessarily proportional to the unit quaternions of some basis, although they could be. Nevertheless, this means that regardless of how many terms exist in a given bilinear expression for h, they can always be reduced to represent h in at most 16 such terms; $\ddot{h}_n \mapsto \ddot{h}_{16}$. But, the convenient format may have us treating expressions with $n > 16$ to avoid decomposing the original quaternion parameters into their components just to effect a reduction in term count.

We now have explicit solutions, $\ddot{h}_1^{-1}$, $\ddot{h}_2^{-1}$, and, $\ddot{h}_3^{-1}$. Then, $\ddot{h}(A,B)_n^{-1}$ and $\ddot{h}(P,Q)_n^{-1}$, would represent two different inverses with the same number of terms n, but differing sequence parameters, A_k, B_k and P_j, Q_j, with $j, k = 1, 2, \ldots, n$; the $\ddot{h}_n \equiv \ddot{h}(A,B)_n$ representing the corresponding two-hand expression that is, or is to be, inverted. We shall use this symbolic notation in the sections below, where we solve systems of linear equations.

Simpler Method

Using the right and left conjugate operators, $(\,\cdot\,)^{*R}$ and $(\,\cdot\,)^{*L}$, we now work out the solution to the linear problem in a more direct manner. This method parallels how we think about the idea of the conjugate in the more usual one-hand algebra of Hamilton's quaternion calculus.

LET, $h \equiv \ddot{h}_n,\ n \leq 2:$

$$\begin{aligned} &[1]\quad h\hat{q} = \hat{C} && [4]\quad h^{-1} = \frac{(h^{*R}h)^{*L}h^{*R}}{(h^{*R}h)^{*L}h^{*R}h} \\ &[2]\quad h^{*R}h\hat{q} = h^{*R}\hat{C} && \\ &[3]\quad (h^{*R}h)^{*L}h^{*R}h\hat{q} = (h^{*R}h)^{*L}h^{*R}\hat{C} && [5]\quad \hat{q} = \left(\frac{(h^{*R}h)^{*L}h^{*R}}{(h^{*R}h)^{*L}h^{*R}h}\right)\hat{C} \end{aligned} \tag{102}$$

e.g.:

$$A_1qB_1 + A_2qB_2 = C \tag{2}$$

$$\begin{aligned}
h &= A_1B_1' + A_2B_2' \\
h^{*R} &= A_1^*B_1' + A_2^*B_2' \\
h^{*R}h &= (A_1^*B_1' + A_2^*B_2')(A_1B_1' + A_2B_2') \\
&= A_1^*B_1'A_1B_1' + A_1^*B_1'A_2B_2' + A_2^*B_2'A_1B_1' + A_2^*B_2'A_2B_2' \\
&= A_1^*A_1B_1'B_1' + A_1^*A_2B_1'B_2' + A_2^*A_1B_2'B_1' + A_2^*A_2B_2'B_2' \\
(h^{*R}h)^{*L} &= A_1^*A_1B_1'^*B_1'^* + A_1^*A_2B_2'^*B_1'^* + A_2^*A_1B_1'^*B_2'^* + A_2^*A_2B_2'^*B_2'^* \\
(h^{*R}h)^{*L}h^{*R} &= (A_1^*A_1B_1'^*B_1'^* + A_1^*A_2B_2'^*B_1'^* + A_2^*A_1B_1'^*B_2'^* + A_2^*A_2B_2'^*B_2'^*)(A_1^*B_1' + A_2^*B_2')
\end{aligned}$$

$$\begin{aligned}
&= A_1^*A_1A_1^*B_1'^*B_1'^*B_1' + A_1^*A_2A_1^*B_2'^*B_1'^*B_1' + A_2^*A_1A_1^*B_1'^*B_2'^*B_1' + A_2^*A_2A_1^*B_2'^*B_2'^*B_1' \\
&+ A_1^*A_1A_2^*B_1'^*B_1'^*B_2' + A_1^*A_2A_2^*B_2'^*B_1'^*B_2' + A_2^*A_1A_2^*B_1'^*B_2'^*B_2' + A_2^*A_2A_2^*B_2'^*B_2'^*B_2'
\end{aligned}$$

$\Longrightarrow$ re-arranging conjugate * factor orders...

$$\begin{aligned}
&= A_1^*A_1A_1^*B_1'^*B_1'B_1'^* + A_1^*A_2A_1^*B_2'^*B_1'B_1'^* + \boldsymbol{A_1^*A_1A_2^*}B_1'^*B_2'^*B_1' + A_2^*A_2A_1^*B_2'^*B_2'^*B_1' \\
&+ A_1^*A_1A_2^*B_1'^*B_1'^*B_2' + \boldsymbol{A_2^*A_2A_1^*}B_2'^*B_1'^*B_2' + A_2^*A_1A_2^*B_1'^*B_2'B_2'^* + A_2^*A_2A_2^*B_2'^*B_2'B_2'^*
\end{aligned}$$

$\Longrightarrow$ $A_1^*A_1A_2^*B_1'^*B_2'^*B_1' + A_1^*A_1A_2^*B_1'^*B_1'^*B_2' = A_1^*A_1A_2^*B_1'^*(B_2'^*B_1' + B_1'^*B_2')$

$\Longrightarrow$ $= A_1^*A_1A_2^*(B_2'^*B_1' + B_1'^*B_2')B_1'^* = A_1^*A_1A_2^*B_2'^*B_1'B_1'^* + A_1^*A_1A_2^*B_1'^*B_2'B_1'^*$ etc...

$$\begin{aligned}
\therefore\quad &= A_1^*A_1A_1^*B_1'^*B_1'B_1'^* + A_1^*A_2A_1^*B_2'^*B_1'B_1'^* + A_1^*A_1A_2^*B_2'^*B_1'B_1'^* + A_2^*A_2A_1^*B_2'^*B_1'B_2'^* \\
&+ A_1^*A_1A_2^*B_1'^*B_2'B_1'^* + A_2^*A_2A_1^*B_1'^*B_2'B_2'^* + A_2^*A_1A_2^*B_1'^*B_2'B_2'^* + A_2^*A_2A_2^*B_2'^*B_2'B_2'^*
\end{aligned}$$

$\Longrightarrow$ so all terms now have format $A^*AA^*B^*BB^*$ compare eqn (30)

$$\begin{aligned}
&= |A_1|^2A_1^*B_1'^*|B_1|^2 + A_1^*A_2A_1^*B_2'^*|B_1|^2 + |A_1|^2A_2^*B_2'^*|B_1|^2 + |A_2|^2A_1^*B_2'^*B_1'B_2'^* \\
&+ |A_1|^2A_2^*B_1'^*B_2'B_1'^* + |A_2|^2A_1^*B_1'^*|B_2|^2 + A_2^*A_1A_2^*B_1'^*|B_2|^2 + |A_2|^2A_2^*B_2'^*|B_2|^2
\end{aligned}$$

$$\begin{aligned}
&= (|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2)(A_1^*B_1'^* + A_2^*B_2'^*) + |A_2|^2A_1^*B_2'^*B_1'B_2'^* \\
&+ |A_1|^2A_2^*B_1'^*B_2'B_1'^* + |B_2|^2A_2^*A_1A_2^*B_1'^* + |B_1|^2A_1^*A_2A_1^*B_2'^*
\end{aligned}$$

$$\begin{aligned}
(h^{*R}h)^{*L}h^{*R}h &= (h^{*R}h)^{*L}h^{*R}(A_1B_1' + A_2B_2') \\
&= ((|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2)(A_1^*B_1'^* + A_2^*B_2'^*) + |A_2|^2A_1^*B_2'^*B_1'B_2'^*)(A_1B_1' + A_2B_2') \\
&+ (|A_1|^2A_2^*B_1'^*B_2'B_1'^* + |B_2|^2A_2^*A_1A_2^*B_1'^* + |B_1|^2A_1^*A_2A_1^*B_2'^*)(A_1B_1' + A_2B_2')
\end{aligned}$$

$$
\begin{aligned}
(h^{*R}h)^{*L}h^{*R}h &= (|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)^2\\
&+(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)(A_1^*A_2B_1'^*B_2'+A_2^*A_1B_2'^*B_1')\\
&+(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)(A_1^*A_2B_2'^*B_1'+A_2^*A_1B_1'^*B_2')\\
&+|A_1|^2|A_2|^2(B_1'^*B_2'B_1'^*B_2'+B_2'^*B_1'B_2'^*B_1')+|B_1|^2|B_2|^2(A_1^*A_2A_1^*A_2+A_2^*A_1A_2^*A_1)
\end{aligned}
$$

$$
\begin{aligned}
&\Longrightarrow \quad B_1'^*B_2'B_1'^*B_2'+B_2'^*B_1'B_2'^*B_1'=(B_1'^*B_2')^2+(B_2'^*B_1')^2=(B_1^*B_2)^2+(B_2^*B_1)^2\\
&\Longrightarrow \quad =(B_1^*B_2+B_2^*B_1)^2-2B_1^*B_1B_2^*B_2=(2\beta)^2-2|B_1|^2|B_2|^2 \text{ etc...}
\end{aligned}
$$

$$
\begin{aligned}
&=(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)^2-4|A_1|^2|A_2|^2|B_1|^2|B_2|^2\\
&+4(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)\alpha\beta+|A_1|^2|A_2|^2(2\beta)^2+|B_1|^2|B_2|^2(2\alpha)^2
\end{aligned}
$$

$$
\therefore \qquad h^{-1} = \left(\frac{\begin{array}{c}(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)(A_1^*B_1'^*+A_2^*B_2'^*)+|A_2|^2A_1^*B_2'^*B_1'B_2'^*\\ +|A_1|^2A_2^*B_1'^*B_2'B_1'^*+|B_2|^2A_2^*A_1A_2^*B_1'^*+|B_1|^2A_1^*A_2A_1^*B_2'^*\end{array}}{\begin{array}{l}(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)^2-4|A_1|^2|A_2|^2|B_1|^2|B_2|^2\\ +4(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)\alpha\beta+|A_1|^2|A_2|^2(2\beta)^2+|B_1|^2|B_2|^2(2\alpha)^2\end{array}}\right)
$$

$$
q = \left(\frac{\begin{array}{c}(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)(A_1^*CB_1^*+A_2^*CB_2^*)+|A_2|^2A_1^*CB_2^*B_1B_2^*\\ +|A_1|^2A_2^*CB_1^*B_2B_1^*+|B_2|^2A_2^*A_1A_2^*CB_1^*+|B_1|^2A_1^*A_2A_1^*CB_2^*\end{array}}{\begin{array}{l}(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)^2-4|A_1|^2|A_2|^2|B_1|^2|B_2|^2\\ +4(|A_1|^2|B_1|^2+|A_2|^2|B_2|^2)\alpha\beta+|A_1|^2|A_2|^2(2\beta)^2+|B_1|^2|B_2|^2(2\alpha)^2\end{array}}\right)
$$

where, $\quad 2\alpha = A_1^*A_2+A_2^*A_1 \;, \quad 2\beta = B_1^*B_2+B_2^*B_1$

An alternative parallel method, that begins with the left conjugate, h^{*L} , as first factor, and so builds up the numerator, $(h^{*L}h)^{*R}h^{*L}$, instead of, $(h^{*R}h)^{*L}h^{*R}$, solves this problem again. The reader may verify the result is the same.

The first of the linear equations, (1), could also be trivially solved with this method.
e.g:

$$
\begin{aligned}
A_1qB_1 &= C \qquad\qquad (1)\\
h &= A_1B_1'\\
h^{*R} &= A_1^*B_1'\\
h^{*R}h &= (A_1^*B_1')(A_1B_1')\\
&= A_1^*A_1B_1'B_1' = |A_1|^2(B_1')^2\\
(h^{*R}h)^{*L} &= |A_1|^2(B_1'^*)^2\\
(h^{*R}h)^{*L}h^{*R} &= |A_1|^2(B_1'^*)^2A_1^*B_1' = |A_1|^2|B_1|^2A_1^*B_1'^*\\
(h^{*R}h)^{*L}h^{*R}h &= |A_1|^2|B_1|^2A_1^*B_1'^*(A_1B_1') = |A_1|^4|B_1|^4
\end{aligned}
$$

$$
\therefore \quad h^{-1}=\frac{|A_1|^2|B_1|^2A_1^*B_1'^*}{|A_1|^4|B_1|^4}=\frac{A_1^*B_1'^*}{|A_1|^2|B_1|^2} \qquad \text{and,} \qquad q = \frac{A_1^*CB_1^*}{|A_1|^2|B_1|^2}
$$

General Solution

The two conjugated cubic forms, $(h^{*R}h)^{*L}h^{*R}$ and $(h^{*L}h)^{*R}h^{*L}$, which both independently solve the 1-term and 2-term problems, as illustrated above, now need to be combined together with a third conjugated cube, like $h^{*R}hh^{*L}$, in order to obtain the solutions for the 3-term and higher linear problems. The method and solution is discussed in the APPENDIX: CONJUGATED CUBES, where a formula for h^{-1} is given in (A-35). We had intended to present that solution, based on these three cubic forms, in this section, but have since discovered a more efficient formula, which we present here instead. The new "Gilgamesh Solution" is also discussed in the appendix and given there in (A-60).

LET, $h \equiv \ddot{h}_n,\ n \geq 1:$

$$
\begin{array}{ll}
[1] & h\hat{q} = \hat{C} \\
[2] & (hh^*)^{*R}h\hat{q} = (hh^*)^{*R}\hat{C} \\
[3] & h^*(hh^*)^{*R}h\hat{q} = h^*(hh^*)^{*R}\hat{C} \\
[4] & (h^{*L}(h^{*R}h^{*L})^{*R})^{*R}h\hat{q} = (h^{*L}(h^{*R}h^{*L})^{*R})^{*R}\hat{C} \\
[5] & (h^*(hh^*)^{*R} + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R})h\hat{q} = (h^*(hh^*)^{*R} + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R})\hat{C} \\
[6] & h^{-1} = \dfrac{h^*(hh^*)^{*R} + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}}{h^*(hh^*)^{*R}h + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}h} \\
[7] & \hat{q} = \left(\dfrac{h^*(hh^*)^{*R} + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}}{h^*(hh^*)^{*R}h + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}h}\right)\hat{C}
\end{array}
\tag{103}
$$

We can no longer just multiply the L-H-S of the linear equation by simple conjugated factors to reduce h to scalar, when we deal with arbitrary n. We must also combine these products to obtain solutions. Two new cubes, $h^*(hh^*)^{*R}$ and $(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}$, are now built up on the L-H-S to produce eqns [3] and [4] above, respectively, and then the former is added to twice the latter to obtain the reduction to scalar in [5]. The inverse, h^{-1}, and solution for q, follows. The Gilgamesh Solution was found by symbolic computation, and is non-obvious; there is no known intuitive analytical derivation, so let us verify it by analytical hand reduction for the "three term" problem for which we already know the solution. Because the factor of $1/3$ has been removed from the numerator and denominator of the formula for h^{-1} here (see APPENDIX), this numerator is a factor of 3 greater than the numerator, $(gh)^{*L}g$, of eqn (85). Our obective then, is to prove,

$$
3\cdot(gh)^{*L}g \;\equiv\; 1\cdot h^*(hh^*)^{*R} \;+\; 2\cdot(h^{*L}(h^{*R}h^{*L})^{*R})^{*R} \qquad \text{FOR,}\quad h \equiv \ddot{h}_3 \tag{104}
$$

Once the numerators have been proven equal, the denominators are automatically then the same, since, in both cases, the denominator is just the factor h times the numerator. Then, it follows, the inverse, h^{-1}, and the solution, q, must be equivalent also. So, let's prove this identity, by hand. First we compute the four conjugated states of h,

$$
\begin{array}{lcccccc}
h & = & A_1B'_1 & + & A_2B'_2 & + & A_3B'_3 \\
h^* & = & A_1^*B'^*_1 & + & A_2^*B'^*_2 & + & A_3^*B'^*_3 \\
h^{*R} & = & A_1^*B'_1 & + & A_2^*B'_2 & + & A_3^*B'_3 \\
h^{*L} & = & A_1B'^*_1 & + & A_2B'^*_2 & + & A_3B'^*_3
\end{array}
\tag{105}
$$

Then, we compute the four intermediate conjugated squares $\sim hh$,

$$
\begin{array}{lcl}
hh^* & = & (A_1B'_1 + A_2B'_2 + A_3B'_3)(A_1^*B'^*_1 + A_2^*B'^*_2 + A_3^*B'^*_3) \\
(hh^*)^{*R} & = & ((A_1B'_1 + A_2B'_2 + A_3B'_3)(A_1^*B'^*_1 + A_2^*B'^*_2 + A_3^*B'^*_3))^{*R} \\
h^{*R}h^{*L} & = & (A_1^*B'_1 + A_2^*B'_2 + A_3^*B'_3)(A_1B'^*_1 + A_2B'^*_2 + A_3B'^*_3) \\
(h^{*R}h^{*L})^{*R} & = & ((A_1^*B'_1 + A_2^*B'_2 + A_3^*B'_3)(A_1B'^*_1 + A_2B'^*_2 + A_3B'^*_3))^{*R}
\end{array}
\tag{106}
$$

$$
\begin{array}{llll}
hh^* = & (hh^*)^{*R} = & h^{*R}h^{*L} = & (h^{*R}h^{*L})^{*R} = \\
+A_1\cdot A_1^*\cdot B'_1\cdot B'^*_1 & +A_1\cdot A_1^*\cdot B'_1\cdot B'^*_1 & +A_1^*\cdot A_1\cdot B'_1\cdot B'^*_1 & +A_1^*\cdot A_1\cdot B'_1\cdot B'^*_1 \\
+A_1\cdot A_2^*\cdot B'_1\cdot B'^*_2 & +A_2\cdot A_1^*\cdot B'_1\cdot B'^*_2 & +A_1^*\cdot A_2\cdot B'_1\cdot B'^*_2 & +A_2^*\cdot A_1\cdot B'_1\cdot B'^*_2 \\
+A_1\cdot A_3^*\cdot B'_1\cdot B'^*_3 & +A_3\cdot A_1^*\cdot B'_1\cdot B'^*_3 & +A_1^*\cdot A_3\cdot B'_1\cdot B'^*_3 & +A_3^*\cdot A_1\cdot B'_1\cdot B'^*_3 \\
+A_2\cdot A_1^*\cdot B'_2\cdot B'^*_1 & +A_1\cdot A_2^*\cdot B'_2\cdot B'^*_1 & +A_2^*\cdot A_1\cdot B'_2\cdot B'^*_1 & +A_1^*\cdot A_2\cdot B'_2\cdot B'^*_1 \\
+A_2\cdot A_2^*\cdot B'_2\cdot B'^*_2 & +A_2\cdot A_2^*\cdot B'_2\cdot B'^*_2 & +A_2^*\cdot A_2\cdot B'_2\cdot B'^*_2 & +A_2^*\cdot A_2\cdot B'_2\cdot B'^*_2 \\
+A_2\cdot A_3^*\cdot B'_2\cdot B'^*_3 & +A_3\cdot A_2^*\cdot B'_2\cdot B'^*_3 & +A_2^*\cdot A_3\cdot B'_2\cdot B'^*_3 & +A_3^*\cdot A_2\cdot B'_2\cdot B'^*_3 \\
+A_3\cdot A_1^*\cdot B'_3\cdot B'^*_1 & +A_1\cdot A_3^*\cdot B'_3\cdot B'^*_1 & +A_3^*\cdot A_1\cdot B'_3\cdot B'^*_1 & +A_1^*\cdot A_3\cdot B'_3\cdot B'^*_1 \\
+A_3\cdot A_2^*\cdot B'_3\cdot B'^*_2 & +A_2\cdot A_3^*\cdot B'_3\cdot B'^*_2 & +A_3^*\cdot A_2\cdot B'_3\cdot B'^*_2 & +A_2^*\cdot A_3\cdot B'_3\cdot B'^*_2 \\
+A_3\cdot A_3^*\cdot B'_3\cdot B'^*_3 & +A_3\cdot A_3^*\cdot B'_3\cdot B'^*_3 & +A_3^*\cdot A_3\cdot B'_3\cdot B'^*_3 & +A_3^*\cdot A_3\cdot B'_3\cdot B'^*_3
\end{array}
\tag{107}
$$

Now, we compute the final conjugated cubes $\sim hhh$,

$$
\begin{array}{lll}
h^*(hh^*)^{*R} = & (h^{*L}(h^{*R}h^{*L})^{*R})^{*R} = & h^\dagger = \\
+A_1^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_1'^{*} & +A_1^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_1'^{*} & +A_1^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_1'^{*} \\
+A_1^* \cdot A_1 \cdot A_2^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_1 \cdot A_2^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_1 \cdot A_2^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} \\
+A_1^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} \\
+A_1^* \cdot A_2 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_2 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_2 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_2'^{*} \\
+A_1^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_1' \cdot B_2'^{*} & +A_1^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_1' \cdot B_2'^{*} \\
+A_1^* \cdot A_2 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} & +\boldsymbol{A_1^* \cdot A_2 \cdot A_3^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*}} & +A_1^* \cdot A_2 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} \\
+A_1^* \cdot A_3 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_3 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_3 \cdot A_1^* \cdot B_1'^{*} \cdot B_1' \cdot B_3'^{*} \\
+\boldsymbol{A_1^* \cdot A_3 \cdot A_2^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*}} & +\boldsymbol{A_1^* \cdot A_3 \cdot A_2^* \cdot B_2'^{*} \cdot B_1' \cdot B_3'^{*}} & +A_1^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*} \\
+A_1^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_1' \cdot B_3'^{*} & +A_1^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_1' \cdot B_3'^{*} \\
+A_2^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_2' \cdot B_1'^{*} \\
+A_2^* \cdot A_1 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_1 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_1 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} \\
+\boldsymbol{A_2^* \cdot A_1 \cdot A_3^* \cdot B_2'^{*} \cdot B_3' \cdot B_1'^{*}} & +\boldsymbol{A_2^* \cdot A_1 \cdot A_3^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*}} & +A_2^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_1'^{*} \\
+A_2^* \cdot A_2 \cdot A_1^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_2 \cdot A_1^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} & +A_2^* \cdot A_2 \cdot A_1^* \cdot B_2'^{*} \cdot B_2' \cdot B_1'^{*} \\
+A_2^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_2'^{*} & +A_2^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_2'^{*} & +A_2^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_2' \cdot B_2'^{*} \\
+A_2^* \cdot A_2 \cdot A_3^* \cdot B_2'^{*} \cdot B_2' \cdot B_3'^{*} & +A_2^* \cdot A_2 \cdot A_3^* \cdot B_2'^{*} \cdot B_2' \cdot B_3'^{*} & +A_2^* \cdot A_2 \cdot A_3^* \cdot B_2'^{*} \cdot B_2' \cdot B_3'^{*} \\
+\boldsymbol{A_2^* \cdot A_3 \cdot A_1^* \cdot B_2'^{*} \cdot B_1' \cdot B_3'^{*}} & +\boldsymbol{A_2^* \cdot A_3 \cdot A_1^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*}} & +A_2^* \cdot A_3 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} \\
+A_2^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_2'^{*} & +A_2^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_2'^{*} & +A_2^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_2'^{*} \\
+A_2^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_2' \cdot B_3'^{*} & +A_2^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_2' \cdot B_3'^{*} & +A_2^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_2' \cdot B_3'^{*} \\
+A_3^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_3' \cdot B_1'^{*} & +A_3^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_3' \cdot B_1'^{*} & +A_3^* \cdot A_1 \cdot A_1^* \cdot B_1'^{*} \cdot B_3' \cdot B_1'^{*} \\
+\boldsymbol{A_3^* \cdot A_1 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*}} & +\boldsymbol{A_3^* \cdot A_1 \cdot A_2^* \cdot B_2'^{*} \cdot B_3' \cdot B_1'^{*}} & +A_3^* \cdot A_1 \cdot A_2^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*} \\
+A_3^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_3'^{*} & +A_3^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_3'^{*} & +A_3^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_3'^{*} \\
+A_3^* \cdot A_2 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} & +\boldsymbol{A_3^* \cdot A_2 \cdot A_1^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*}} & +A_3^* \cdot A_2 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} \\
+A_3^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_2 \cdot A_2^* \cdot B_2'^{*} \cdot B_3' \cdot B_2'^{*} \\
+A_3^* \cdot A_2 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_2 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_2 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} \\
+A_3^* \cdot A_3 \cdot A_1^* \cdot B_3'^{*} \cdot B_3' \cdot B_1'^{*} & +A_3^* \cdot A_3 \cdot A_1^* \cdot B_3'^{*} \cdot B_3' \cdot B_1'^{*} & +A_3^* \cdot A_3 \cdot A_1^* \cdot B_3'^{*} \cdot B_3' \cdot B_1'^{*} \\
+A_3^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} & +A_3^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_3' \cdot B_2'^{*} \\
+A_3^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_3'^{*} & +A_3^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_3'^{*} & +A_3^* \cdot A_3 \cdot A_3^* \cdot B_3'^{*} \cdot B_3' \cdot B_3'^{*}
\end{array}
\tag{108}
$$

The first two columns show the two final cubes. The intermediate cube, $h^{*L}(h^{*R}h^{*L})^{*R}$, is not shown, but the reader is invited to compute this cube and verify for himself that, when right conjugated, $(\cdot)^{*R}$, it yields the results in the middle column. The column on the right is the adjoint in the form given in eqn (85), i.e. $h^\dagger = (gh)^{*L}g$; all three columns are put in the same conventional "standard order" we previously defined there, so that they may be easily compared. Each row in the table shows, therefore, the related bi-cubic terms, $A^*_{\cdot}A_{\cdot}A^*_{\cdot}B'^*_{\cdot}B'_{\cdot}B'^*_{\cdot}$, for each of the two cubes and the adjoint. A review of the table entries reveals that all but 6 of the 27 rows have identical bi-cubic terms in them. This means that adding a given term from the first cube to twice that from the second cube will indeed give three times the term in the adjoint, for each row in most cases. On a term by term basis, therefore, most of the rows verify the identity in (104). The bi-cubic entries shown in boldface text are the terms that differ from that in the adjoint. So, we must separate these terms and treat them as group to prove the identity. These six rows happen to contain the same group of six bi-cubic terms that form the last expression block in the solution for $(gh)^{*L}g$ given in eqn (88). These are the terms with all three subscript indicies unique. The terms with two or three repeated indices are identical in the two cubes and the adjoint. The easiest way to prove the identity (104), therefore, is to subtract 3 times the adjoint from the sum of the first cube and twice the second cube, which reduces the problem to that of proving that the following expression block reduces to zero;

$$
\Delta = 1 \cdot h^*(hh^*)^{*R} + 2 \cdot (h^{*L}(h^{*R}h^{*L})^{*R})^{*R} - 3 \cdot (gh)^{*L}g \quad =
$$

$$
\begin{array}{lclcl}
+1 \cdot A_1^* \cdot A_2 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} & + & 2 \cdot A_1^* \cdot A_2 \cdot A_3^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} & - & 3 \cdot A_1^* \cdot A_2 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} \\
+1 \cdot A_1^* \cdot A_3 \cdot A_2^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*} & + & 2 \cdot A_1^* \cdot A_3 \cdot A_2^* \cdot B_2'^{*} \cdot B_1' \cdot B_3'^{*} & - & 3 \cdot A_1^* \cdot A_3 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*} \\
+1 \cdot A_2^* \cdot A_1 \cdot A_3^* \cdot B_2'^{*} \cdot B_3' \cdot B_1'^{*} & + & 2 \cdot A_2^* \cdot A_1 \cdot A_3^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*} & - & 3 \cdot A_2^* \cdot A_1 \cdot A_3^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} \\
+1 \cdot A_2^* \cdot A_3 \cdot A_1^* \cdot B_2'^{*} \cdot B_1' \cdot B_3'^{*} & + & 2 \cdot A_2^* \cdot A_3 \cdot A_1^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*} & - & 3 \cdot A_2^* \cdot A_3 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} \\
+1 \cdot A_3^* \cdot A_1 \cdot A_2^* \cdot B_3'^{*} \cdot B_2' \cdot B_1'^{*} & + & 2 \cdot A_3^* \cdot A_1 \cdot A_2^* \cdot B_2'^{*} \cdot B_3' \cdot B_1'^{*} & - & 3 \cdot A_3^* \cdot A_1 \cdot A_2^* \cdot B_1'^{*} \cdot B_2' \cdot B_3'^{*} \\
+1 \cdot A_3^* \cdot A_2 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*} & + & 2 \cdot A_3^* \cdot A_2 \cdot A_1^* \cdot B_1'^{*} \cdot B_3' \cdot B_2'^{*} & - & 3 \cdot A_3^* \cdot A_2 \cdot A_1^* \cdot B_3'^{*} \cdot B_1' \cdot B_2'^{*}
\end{array}
\tag{109}
$$

To reduce this expression block, we first expand the right and left cubic quaternion factors in other terms, so that the highest order—i.e. cubic—term is the same, $A_1^*A_2A_3^*$ or $B_1'^*B_2'B_3'^*$, for all six factor permutations.

$$
\begin{aligned}
A_1^*A_2A_3^* &= A_1^*A_2A_3^* \\
A_1^*A_3A_2^* &= A_1^*(2\alpha_1 - A_2A_3^*) = A_1^*2\alpha_1 - A_1^*A_2A_3^* \\
A_2^*A_1A_3^* &= (2\alpha_3 - A_1^*A_2)A_3^* = 2\alpha_3A_3^* - A_1^*A_2A_3^* \\
A_2^*A_3A_1^* &= A_2^*(2\alpha_2 - A_1A_3^*) = A_2^*2\alpha_2 - A_2^*A_1A_3^* = A_2^*2\alpha_2 - (2\alpha_3 - A_1^*A_2)A_3^* = A_2^*2\alpha_2 - 2\alpha_3A_3^* + A_1^*A_2A_3^* \\
A_3^*A_1A_2^* &= (2\alpha_2 - A_1^*A_3)A_2^* = 2\alpha_2A_2^* - A_1^*A_3A_2^* = 2\alpha_2A_2^* - A_1^*(2\alpha_1 - A_2A_3^*) = 2\alpha_2A_2^* - A_1^*2\alpha_1 + A_1^*A_2A_3^* \\
A_3^*A_2A_1^* &= (2\alpha_1 - A_2^*A_3)A_1^* = 2\alpha_1A_1^* - A_2^*A_3A_1^* = 2\alpha_1A_1^* - A_2^*(2\alpha_2 - A_1A_3^*) = 2\alpha_1A_1^* - A_2^*2\alpha_2 + A_2^*A_1A_3^* \\
&= 2\alpha_1A_1^* - A_2^*2\alpha_2 + (2\alpha_3 - A_1^*A_2)A_3^* = (2\alpha_1A_1^* - A_2^*2\alpha_2 + 2\alpha_3A_3^* - A_1^*A_2A_3^*)
\end{aligned}
\tag{110}
$$

$$
\begin{aligned}
B_1'^*B_2'B_3'^* &= B_1'^*B_2'B_3'^* \\
B_1'^*B_3'B_2'^* &= B_1'^*(2\beta_1 - B_2'B_3'^*) = B_1'^*2\beta_1 - B_1'^*B_2'B_3'^* \\
B_2'^*B_1'B_3'^* &= (2\beta_3 - B_1'^*B_2')B_3'^* = 2\beta_3B_3'^* - B_1'^*B_2'B_3'^* \\
B_2'^*B_3'B_1'^* &= B_2'^*(2\beta_2 - B_1'B_3'^*) = B_2'^*2\beta_2 - B_2'^*B_1'B_3'^* = B_2'^*2\beta_2 - (2\beta_3 - B_1'^*B_2')B_3'^* = B_2'^*2\beta_2 - 2\beta_3B_3'^* + B_1'^*B_2'B_3'^* \\
B_3'^*B_1'B_2'^* &= (2\beta_2 - B_1'^*B_3')B_2'^* = 2\beta_2B_2'^* - B_1'^*B_3'B_2'^* = 2\beta_2B_2'^* - B_1'^*(2\beta_1 - B_2'B_3'^*) = 2\beta_2B_2'^* - B_1'^*2\beta_1 + B_1'^*B_2'B_3'^* \\
B_3'^*B_2'B_1'^* &= (2\beta_1 - B_2'^*B_3')B_1'^* = 2\beta_1B_1'^* - B_2'^*B_3'B_1'^* = 2\beta_1B_1'^* - B_2'^*(2\beta_2 - B_1'B_3'^*) = 2\beta_1B_1'^* - B_2'^*2\beta_2 + B_2'^*B_1'B_3'^* \\
&= 2\beta_1B_1'^* - B_2'^*2\beta_2 + (2\beta_3 - B_1'^*B_2')B_3'^* = 2\beta_1B_1'^* - B_2'^*2\beta_2 + 2\beta_3B_3'^* - B_1'^*B_2'B_3'^*
\end{aligned}
\tag{111}
$$

Then the results for the A-cubes, A^*AA^*, are substituted, and the expression block reduced and re-arranged; we obtain;

$$
\Delta \quad = \quad
\begin{array}{rcrcrcr}
(-2\cdot A_1^*A_2A_3^* & + & 4\cdot 2\alpha_1A_1^* & - & 1\cdot 2\alpha_2A_2^* & - & 2\cdot 2\alpha_3A_3^*)B_1'^*B_2'B_3'^* \\
+(-1\cdot A_1^*A_2A_3^* & + & 2\cdot 2\alpha_1A_1^* & - & 2\cdot 2\alpha_2A_2^* & - & 1\cdot 2\alpha_3A_3^*)B_1'^*B_3'B_2'^* \\
+(-1\cdot A_1^*A_2A_3^* & + & 2\cdot 2\alpha_1A_1^* & + & 1\cdot 2\alpha_2A_2^* & - & 1\cdot 2\alpha_3A_3^*)B_2'^*B_1'B_3'^* \\
+(+1\cdot A_1^*A_2A_3^* & - & 2\cdot 2\alpha_1A_1^* & + & 2\cdot 2\alpha_2A_2^* & + & 1\cdot 2\alpha_3A_3^*)B_2'^*B_3'B_1'^* \\
+(+1\cdot A_1^*A_2A_3^* & - & 2\cdot 2\alpha_1A_1^* & - & 1\cdot 2\alpha_2A_2^* & + & 1\cdot 2\alpha_3A_3^*)B_3'^*B_1'B_2'^* \\
+(+2\cdot A_1^*A_2A_3^* & - & 4\cdot 2\alpha_1A_1^* & + & 1\cdot 2\alpha_2A_2^* & + & 2\cdot 2\alpha_3A_3^*)B_3'^*B_2'B_1'^*
\end{array}
\tag{112}
$$

Finally, the results for the B'-cubes, $B'^*B'B'^*$, are substituted, and the expression block simplifies to zero;

$$
\begin{aligned}
\Delta \quad = \quad & \begin{array}{l}
+1\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^* \\
+1\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^* \\
+1\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^* \\
+1\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot A_1^*A_2A_3^*B_1'^*B_2'B_3'^*
\end{array} \\[1ex]
& \begin{array}{lll}
+2\cdot A_1^*A_2A_3^*2\beta_1B_1'^* & +1\cdot A_1^*A_2A_3^*2\beta_2B_2'^* & +2\cdot A_1^*A_2A_3^*2\beta_3B_3'^* \\
-1\cdot A_1^*A_2A_3^*B_1'^*2\beta_1 & +1\cdot A_1^*A_2A_3^*B_2'^*2\beta_2 & -1\cdot A_1^*A_2A_3^*2\beta_3B_3'^* \\
-1\cdot A_1^*A_2A_3^*B_1'^*2\beta_1 & -2\cdot A_1^*A_2A_3^*B_2'^*2\beta_2 & -1\cdot A_1^*A_2A_3^*2\beta_3B_3'^*
\end{array} \\[1ex]
& \begin{array}{lll}
+4\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & +2\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & +1\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^* \\
+4\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & +2\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & +1\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & -1\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & +1\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & -1\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & +1\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & -1\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & -2\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^* \\
-2\cdot 2\alpha_1A_1^*B_1'^*B_2'B_3'^* & -1\cdot 2\alpha_2A_2^*B_1'^*B_2'B_3'^* & -2\cdot 2\alpha_3A_3^*B_1'^*B_2'B_3'^*
\end{array} \\[1ex]
& \begin{array}{lll}
+2\cdot 2\alpha_1A_1^*B_1'^*2\beta_1 & +1\cdot 2\alpha_2A_2^*2\beta_1B_1'^* & +2\cdot 2\alpha_3A_3^*2\beta_1B_1'^* \\
+2\cdot 2\alpha_1A_1^*B_1'^*2\beta_1 & +1\cdot 2\alpha_2A_2^*B_1'^*2\beta_1 & -1\cdot 2\alpha_3A_3^*B_1'^*2\beta_1 \\
-4\cdot 2\alpha_1A_1^*2\beta_1B_1'^* & -2\cdot 2\alpha_2A_2^*B_1'^*2\beta_1 & -1\cdot 2\alpha_3A_3^*B_1'^*2\beta_1 \\
+4\cdot 2\alpha_1A_1^*B_2'^*2\beta_2 & -1\cdot 2\alpha_2A_2^*2\beta_2B_2'^* & +1\cdot 2\alpha_3A_3^*2\beta_2B_2'^* \\
-2\cdot 2\alpha_1A_1^*2\beta_2B_2'^* & -1\cdot 2\alpha_2A_2^*B_2'^*2\beta_2 & +1\cdot 2\alpha_3A_3^*B_2'^*2\beta_2 \\
-2\cdot 2\alpha_1A_1^*B_2'^*2\beta_2 & +2\cdot 2\alpha_2A_2^*B_2'^*2\beta_2 & -2\cdot 2\alpha_3A_3^*B_2'^*2\beta_2 \\
+2\cdot 2\alpha_1A_1^*2\beta_3B_3'^* & +1\cdot 2\alpha_2A_2^*2\beta_3B_3'^* & +2\cdot 2\alpha_3A_3^*2\beta_3B_3'^* \\
+2\cdot 2\alpha_1A_1^*2\beta_3B_3'^* & +1\cdot 2\alpha_2A_2^*2\beta_3B_3'^* & -1\cdot 2\alpha_3A_3^*2\beta_3B_3'^* \\
-4\cdot 2\alpha_1A_1^*2\beta_3B_3'^* & -2\cdot 2\alpha_2A_2^*2\beta_3B_3'^* & -1\cdot 2\alpha_3A_3^*2\beta_3B_3'^*
\end{array} \\[1ex]
= \quad & 0 \qquad\qquad \text{Q.E.D.}
\end{aligned}
\tag{113}
$$

This verifies that the "Gilgamesh Solution" produces the same result for the " three term " problem, as that which we found previously by our initial alternate method. But, the new method solves the n-term linear problem also, and is thus the general solution to the arbitrary linear problem. This is the "*general formula for all possible unique solutions*" to the n-term linear problem. When no general unique solution exists, there may be several special case solutions, but these are *singular solutions*, and so are not given by this method. However, what is also given by the Gilgamesh formula, is the necessary conditions for the existance of a general unique solution for any given n-term problem. That this is so, follows immediately from *equivalence of the hexpentaquaternion algebra to ordinary matrix algebra*. This condition, *expressed entirely in quaternion variables*, is that the determinant of H, the matrix form of h, is non-zero; $det(H) \neq 0$. Previous art gave this necessary condition formulated with the components of the quaternions, but not in the quaternions themselves! As discussed in the APPENDIX , *the Gilgamesh solution is obtained by constructing a quaternion expansion of* $H^\dagger$, *the adjoint of the matrix* H, *and then dividing by the quaternion expansion of the determinant*, $det(H)$. These particular quaternion expansions are given in (A-53) and (A-54), and repeated here;

$$H^\dagger = \frac{1}{3}\cdot H^*(HH^*)^{*R} + \frac{2}{3}\cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R} \tag{114}$$

$$det(H) = \frac{1}{3}\cdot H^*(HH^*)^{*R}H + \frac{2}{3}\cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R}H \tag{115}$$

With these constructions, when the $h \equiv H$ is in bilinear form, we can substitute the $H = \sum A_i B'_j$ to obtain the expressions for the adjoint and determinant in whole quaternions. The components of the quaternions are never referenced at all. The two cubes are expanded by the forms given in (A-57) and (A-58), and allow us to write the adjoint and determinant in the original A_k and B_k quaternion parameters,

$$H^\dagger = \frac{1}{3}\cdot \sum \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) B_i'^* B_j' B_k'^* \tag{116}$$

$$det(H) = \frac{1}{3}\cdot \sum \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) A_l B_i'^* B_j' B_k'^* B_l' \tag{117}$$

At first glance, the adjoint appears to have $3n^3$ terms of the bi-cubic form $A_.^* A_. A_.^* B_.'^* B_.' B_.'^*$; n^3 contributed by the first cube, and $2n^3$ contributed by the second cube. However, a re-arrangment shows that these terms can all be re-written $3 \cdot A_.^* A_. A_.^* B_.'^* B_.' B_.'^*$, whence the $1/3$ factor preceeding the summation then reduces the count to exactly n^3. Let us see how this occurs. The permutations of the three ijk indicies can be partitioned into three sets: In the first set, all indicies are the same, $i = j = k$; in the second set, exactly two indicies are equal, $i = j \neq k \mid k = i \neq j \mid j = k \neq i$; and in the third set, all indicies are unique, $i \neq j$ & $j \neq k$ & $k \neq i$. This partitions the summation for the adjoint into three distinct partial sums; let's call them T_1, T_2, T_3, so,

$$H^\dagger = \frac{1}{3}\cdot T_1 + \frac{1}{3}\cdot T_2 + \frac{1}{3}\cdot T_3 \tag{118}$$

$$T_1 = \sum_i \left(A_i^* A_i A_i^* + 2A_i^* A_i A_i^*\right) B_i'^* B_i' B_i'^* \tag{119}$$

$$T_2 = \sum_{i\neq j} \left(\left(A_i^* A_j A_i^* + 2A_i^* A_j A_i^*\right) B_i'^* B_i' B_j'^* + \left(A_i^* A_i A_j^* + 2A_j^* A_i A_i^*\right) B_i'^* B_j' B_i'^* + \left(A_i^* A_j A_j^* + 2A_j^* A_j A_i^*\right) B_i'^* B_j' B_j'^*\right) \tag{120}$$

$$T_3 = \sum_{i\neq j, j\neq k, k\neq i} \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) B_i'^* B_j' B_k'^* \tag{121}$$

The first partial sum, T_1, is readily observed to be reduced to the sum of $3 \cdot A_i^* A_i A_i^* B_i'^* B_i' B_i'^*$, so, with the $1/3$ factor in front, this contributes just n terms of the form $A_i^* A_i A_i^* B_i'^* B_i' B_i'^*$ to the adjoint. The second partial sum, T_2, consists of three parts, each of which can be independently re-written with the 3 factor. The first part is immediately seen to be $3 \cdot A_i^* A_j A_i^* B_i'^* B_i' B_j'^*$. The remaining two parts both contain the scalar $|A_i|^2 = A_i^* A_i = A_i A_i^*$, which therefore commutes with the third A-quaternion, so we may reorder the factors and write $3 \cdot A_i^* A_i A_j^* B_i'^* B_j' B_i'^*$ and $3 \cdot A_i^* A_j A_j^* B_i'^* B_j' B_j'^*$ for these parts. Each of the three parts of T_2 contains $3 \cdot n \cdot (n-1)$ terms, and the $1/3$ factor therefore reduces the contribution to $n(n-1)$ to the adjoint from each part, or a total of $3n(n-1)$ from the complete T_2 summation. These two partial sums can therefore be re-written;

$$T_1 = 3 \cdot \sum_i A_i^* A_i A_i^* B_i'^* B_i' B_i'^* \tag{122}$$

$$T_2 = 3 \cdot \sum_{i\neq j} \left(A_i^* A_j A_i^* B_i'^* B_i' B_j'^* + A_i^* A_i A_j^* B_i'^* B_j' B_i'^* + A_i^* A_j A_j^* B_i'^* B_j' B_j'^*\right) \tag{123}$$

The reduction of the third partial sum, T_3, is the least obvious. But, in fact, we just solved this problem when evaluating the "three term" solution above. This is the sum of unique triple indicies, ijk, and for every particular triple there are six permutations that give rise to 6 terms being contributed from the first cube and 12 terms contributed from the second cube. These 18 terms are "collectively equivalent" to 3 times another different sum of six terms, as shown in the reduction of the $\Delta \to 0$ in eqn (109). The fact that these 18 terms from the "Gilgamesh Solution" are equivalent to 3 times another block of 6 terms, is independent of the indicies chosen. We can replace the 123 indicies with any arbitrary unique ijk triple and re-write the corresponding "Gilgamesh triple" expression block in terms of 3 times the corresponding 6-term block. The 6-term block in (88) encodes the special ordering of the index permutations required to match the unique triple parts of the Gilgamesh Solution. It is useful, therefore, not only for the "three term" solution, but also for the n-term solution as well. So, let us take that block and replace the 123 particular numeric indicies with general ijk indicies, and define V_{ijk} to be this original block of expressions;

$$\begin{aligned} V_{ijk} &= A_i^* \, A_j \, A_k^* \, B_i'^* \, B_k' \, B_j'^* \\ &+ A_i^* \, A_k \, A_j^* \, B_k'^* \, B_j' \, B_i'^* \\ &+ A_j^* \, A_i \, A_k^* \, B_i'^* \, B_k' \, B_j'^* \\ &+ A_j^* \, A_k \, A_i^* \, B_k'^* \, B_i' \, B_j'^* \\ &+ A_k^* \, A_i \, A_j^* \, B_i'^* \, B_j' \, B_k'^* \\ &+ A_k^* \, A_j \, A_i^* \, B_k'^* \, B_i' \, B_j'^* \end{aligned} \tag{124}$$

We then have the following identity,

$$\sum_{\substack{i,j,k \in \{a,b,c\} \\ i \neq j, j \neq k, k \neq i}} \left(A_i^* A_k A_j^* + 2 A_j^* A_k A_i^* \right) B_i'^* B_j' B_k'^* \quad = \quad 3 \cdot V_{abc} \tag{125}$$

On the L-H-S, we have the particular index ordering permutations that define the Gilgamesh unique triple 18-term expression block construction; and on the R-H-S, we have the special index ordering permutation for the 6-term block, revealed by the study of the "three term" solution, that matches this expression block with a factor of 3. The partial sum, T_3, is then seen as the summation of 18-term expression block sums, and we can write,

$$T_3 \quad = \quad 3 \cdot \sum V_{abc} \tag{126}$$

where the $\sum$ is now over the $n!/(3!(n-1)!)$ "unique triples" that make up T_3. Since, all three partial sums, T_1, T_2, and T_3, are reducible to 3 times the sum of bi-cubic terms of the form $A_.^* A_. A_.^* B_.'^* B_.' B_.'^*$, and the adjoint $H^\dagger$ is 1/3 the sum of the partials, it follows that the original $3n^3$ bi-cubic terms that appear in the Gilgamesh formula really contribute just n^3 of these bi-cubic terms to the adjoint. Q.E.D.

The breakdown in the term counts is as follows;

$$\begin{aligned} n(T_1) &= 3 \cdot n \\ n(T_2) &= 3 \cdot n \cdot (n-1) + 3 \cdot n \cdot (n-1) + 3 \cdot n \cdot (n-1) \\ n(T_3) &= 3 \cdot \frac{n!}{3! \cdot (n-3)!} \cdot 6 \\ n(H^\dagger) &= \frac{1}{3} \left(n(T_1) + n(T_2) + n(T_3) \right) \\ &= \frac{1}{3} \cdot \left(3 \cdot n + 3 \cdot n \cdot (n-1) + 3 \cdot n \cdot (n-1) + 3 \cdot n \cdot (n-1) + 3 \cdot \frac{n!}{3! \cdot (n-3)!} \cdot 6 \right) \\ &= n + 3 \cdot n \cdot (n-1) + n \cdot (n-1) \cdot (n-2) \\ &= n + 3n^2 - 3n + n^3 - 3n^2 + 2n \\ &= n^3 \end{aligned} \tag{127}$$

THE DETERMINANT formula, given in (117), contains both right handed and left handed quaternions. But this is just a scalar, $det(H) \in \mathbb{R}$, so we can replace the left handed quaternions with their right handed counterparts. The only rules given in our opening page table that govern these replacements, however, are, $B' + B'^{*} = B + B^{*}$, and, $B'^{*}B' = |B|^2 = B^{*}B$. So, we must show how these lead us to replace the left hand in the more complicated determinant formula, of bi-quartic terms, like $A_j^{*}A_kA_i^{*}A_lB_i'^{*}B_j'B_k'^{*}B_l'$. We must use these elementary replacement rules *to prove* that $det(H)$ is a scalar; for even though we know it's a scalar already, from the analysis of the components in the APPENDIX , we don't want to always have to refer to the components of a quaternion to establish this result. One important observation to make is that, when in the left hand form, the B'-factors commute with the A-factors, but once we convert to right hand, the B-factors no longer commute with the A-factors; the formula composition is then apparently fixed. Yet, because of the flexibility to re-convert the formula back into the dual right hand left hand form again, and move the B'-factors around, the formula must possess a certain amount of re-arrangement symmetry in the exchange of these A and B-factors among themselves. This kind of symmetry is exploited in many of the formulas met above, in the application of the quadratic scalar, $2\beta_{ij} = B_i'^{*}B_j' + B_j'^{*}B_i' = (B_jB_i^{*})' + (B_iB_j^{*})' = B_jB_i^{*} + B_iB_j^{*} = B_i^{*}B_j + B_j^{*}B_i = B_iB_j^{*} + B_jB_i^{*}$ etc.., to reduce and simplify various expressions. We find an index exchange symmetry, $\beta_{ij} = \beta_{ji}$, a hand transformation symmetry, $B'_{.} \rightarrow B_{.}$, $B_{.} \rightarrow B'_{.}$, a conjugation swap symmetry, $B^{*}_{.}B_{.} \rightarrow B_{.}B^{*}_{.}$, and a cyclic permutation symmetry, $B_i^{*}B_j \rightarrow B_jB_i^{*}$, all simultaneously available within the same quadratic construction, leading to several variations on the form of expression. What we need is to analyse the determinant formula in like manner; and we shall therefore make use of these known quadratic symmetries to work out our result.

First we observe that the four $ijkl$ indicies can be partitioned into five disjoint sets; call them, D_1, D_2, D_3, D_4, D_5. In the first set, D_1, all indicies are equal; in the second set, D_2, exactly three indicies are equal; in the third set, D_3, two pairs of indicies are equal; in the fourth set, D_4, exactly one pair of indicies are equal; and in the fifth set, D_5, all indicies are unique. The following table shows the index combination divisions.

$$
\begin{array}{lll}
. & (A_i^{*}A_kA_j^{*} + 2A_j^{*}A_kA_i^{*})\,A_lB_i'^{*}B_j'B_k'^{*}B_l' & i,j,k,l = 1,2,\cdots,n \\
 & & \\
D_1 & (A_i^{*}A_iA_i^{*} + 2A_i^{*}A_iA_i^{*})\,A_iB_i'^{*}B_i'B_i'^{*}B_i' & i = j = k = l \\
 & & \\
D_2 & (A_i^{*}A_iA_i^{*} + 2A_i^{*}A_iA_i^{*})\,A_lB_i'^{*}B_i'B_i'^{*}B_l' & i = j = k \neq l \\
 & (A_i^{*}A_iA_j^{*} + 2A_j^{*}A_iA_i^{*})\,A_iB_i'^{*}B_j'B_i'^{*}B_i' & i = k = l \neq j \\
 & (A_i^{*}A_kA_i^{*} + 2A_i^{*}A_kA_i^{*})\,A_iB_i'^{*}B_i'B_k'^{*}B_i' & i = j = l \neq k \\
 & (A_i^{*}A_jA_j^{*} + 2A_j^{*}A_jA_i^{*})\,A_jB_i'^{*}B_j'B_j'^{*}B_j' & j = k = l \neq i \\
 & & \\
D_3 & (A_i^{*}A_kA_i^{*} + 2A_i^{*}A_kA_i^{*})\,A_kB_i'^{*}B_i'B_k'^{*}B_k' & i = j, k = l, i \neq k \\
 & (A_i^{*}A_jA_j^{*} + 2A_j^{*}A_jA_i^{*})\,A_iB_i'^{*}B_j'B_j'^{*}B_i' & i = l, j = k, i \neq j \\
 & (A_i^{*}A_iA_j^{*} + 2A_j^{*}A_iA_i^{*})\,A_jB_i'^{*}B_j'B_i'^{*}B_j' & i = k, j = l, i \neq j \\
 & & \\
D_4 & (A_i^{*}A_kA_i^{*} + 2A_i^{*}A_kA_i^{*})\,A_lB_i'^{*}B_i'B_k'^{*}B_l' & i = j, i \neq k, i \neq l, k \neq l \\
 & (A_i^{*}A_iA_j^{*} + 2A_j^{*}A_iA_i^{*})\,A_lB_i'^{*}B_j'B_i'^{*}B_l' & i = k, i \neq j, i \neq l, j \neq l \\
 & (A_i^{*}A_kA_j^{*} + 2A_j^{*}A_kA_i^{*})\,A_iB_i'^{*}B_j'B_k'^{*}B_i' & i = l, i \neq j, i \neq k, j \neq k \\
 & (A_i^{*}A_jA_j^{*} + 2A_j^{*}A_jA_i^{*})\,A_lB_i'^{*}B_j'B_j'^{*}B_l' & j = k, i \neq j, i \neq l, j \neq l \\
 & (A_i^{*}A_kA_j^{*} + 2A_j^{*}A_kA_i^{*})\,A_jB_i'^{*}B_j'B_k'^{*}B_j' & j = l, i \neq j, i \neq k, j \neq k \\
 & (A_i^{*}A_kA_j^{*} + 2A_j^{*}A_kA_i^{*})\,A_kB_i'^{*}B_j'B_k'^{*}B_k' & k = l, i \neq j, i \neq k, j \neq k \\
 & & \\
D_5 & (A_i^{*}A_kA_j^{*} + 2A_j^{*}A_kA_i^{*})\,A_lB_i'^{*}B_j'B_k'^{*}B_l' & i \neq j, i \neq k, i \neq l, j \neq k, j \neq l, k \neq l
\end{array}
\qquad (128)
$$

The first set of bi-quartic terms, D_1, is clearly scalar. On a term by term basis, each term has the form, $|A_i|^4|B_i|^4$, and these can immediately be written in the right hand, either $A_i^{*}A_iA_i^{*}A_iB_i^{*}B_iB_i^{*}B_i$, by simply removing the $'$ marks from the B'-factors, or $A_i^{*}A_iA_i^{*}A_i(B_i'^{*}B_i'B_i'^{*}B_i') \rightarrow A_i^{*}A_iA_i^{*}A_i(B_iB_i^{*}B_iB_i^{*})' \rightarrow A_i^{*}A_iA_i^{*}A_iB_iB_i^{*}B_iB_i^{*}$ implementing factor reversal along with the hand change. The D_2 consists of four Gilgamesh type composite bi-quartic terms, which fall into pairs. We put these pairs in consecutive rows for convenience, and re-arrange as follows;

$$\begin{aligned} D_2 &= (A_i^* A_i A_i^* + 2A_i^* A_i A_i^*)\, A_l B_i'^* B_i' B_i'^* B_l' \\ &+ \left(A_i^* A_i A_j^* + 2A_j^* A_i A_i^*\right) A_i B_i'^* B_j' B_i'^* B_i' \\ &+ (A_i^* A_k A_i^* + 2A_i^* A_k A_i^*)\, A_i B_i'^* B_i' B_k'^* B_i' \\ &+ \left(A_i^* A_j A_j^* + 2A_j^* A_j A_i^*\right) A_j B_i'^* B_j' B_j'^* B_j' \end{aligned} \tag{129}$$

$$\begin{aligned} &= (A_i^* A_i + 2A_i^* A_i)\, B_i'^* B_i' \cdot A_i^* A_j B_i'^* B_j' \\ &+ (A_i^* A_i + 2A_i A_i^*)\, B_i'^* B_i' \cdot A_j^* A_i B_i'^* B_j' \\ &+ (A_i^* A_i + 2A_i^* A_i)\, B_i'^* B_i' \cdot A_i^* A_j B_j'^* B_i' \\ &+ (A_i A_i^* + 2A_i^* A_i)\, B_i' B_i'^* \cdot A_j^* A_i B_j'^* B_i' \end{aligned} \tag{130}$$

We re-label the index l to j in the first row, and k to j in the third row, so we only have the two indicies, i, j, in the summation, and we swap the i, j, labels in the fourth row, so we can arrange the prefix A-factors into the same form. We move the $A_.$ quaternions around, using the fact that $A_i^* A_i$ is a scalar, and we gather the similar factors to the left, leaving those that are different from one row to the next on the right. Then we add the first pair of rows and add the second pair of rows to form the quadratic scalars, $A_i^* A_j + A_j^* A_i \in \mathbb{R}$,

$$\begin{aligned} D_2 &= (A_i^* A_i + 2A_i^* A_i)\, B_i' B_i'^* (A_i^* A_j + A_j^* A_i) B_i'^* B_j' \\ &+ (A_i^* A_i + 2A_i^* A_i)\, B_i'^* B_i' (A_i^* A_j + A_j^* A_i) B_j'^* B_i' \end{aligned} \tag{131}$$

$$= (A_i^* A_i + 2A_i^* A_i)\, B_i' B_i'^* (A_i^* A_j + A_j^* A_i)(B_i'^* B_j' + B_j'^* B_i') \tag{132}$$

$$= (A_i^* A_i + 2A_i^* A_i)\, B_i^* B_i (A_i^* A_j + A_j^* A_i)(B_j B_i^* + B_i B_j^*) \tag{133}$$

$$\begin{aligned} &= (A_i^* A_i + 2A_i^* A_i)\, B_i^* B_i (A_i^* A_j + A_j^* A_i) B_j B_i^* \\ &+ (A_i^* A_i + 2A_i^* A_i)\, B_i B_i^* (A_i^* A_j + A_j^* A_i) B_i B_j^* \end{aligned} \tag{134}$$

We add these results together again, to form the final scalar (132), incorporating the B'-factor quadratic scalar, $B_i'^* B_j' + B_j'^* B_i' \in \mathbb{R}$, now, to reveal that the summation of all terms in this second set results in a simple scalar, $D_2 \in \mathbb{R}$. Because of the flexibility provided by the many different equivalent ways to re-arrange the quadratic scalar, we can either permute the factors when removing the prime $'$ marks, i.e. $B_i'^* B_j' + B_j'^* B_i' = (B_j B_i^*)' + (B_i B_j^*)' = B_j B_i^* + B_i B_j^*$, or just remove the primes immediately without reversing the factor order, e.g. $B_i'^* B_j' + B_j'^* B_i' = B_i^* B_j + B_j^* B_i$, and we chose to reverse, because later we shall find that reversal is the only shared option available in all the D_j sets, when we get to the more complicated construction in D_5. The remaining task is to reverse the aggregation steps above, to restore the expression block, as close as possible, to its original composition form, while implementing factor reversal, with these right hand Bs substituting for the previous left hand B's;

$$\begin{aligned} D_2 &= (A_i^* A_i + 2A_i^* A_i)\, B_i B_i^* A_i^* A_j B_j B_i^* \\ &+ (A_i^* A_i + 2A_i A_i^*)\, B_i B_i^* A_j^* A_i B_j B_i^* \\ &+ (A_i^* A_i + 2A_i^* A_i)\, B_i B_i^* A_i^* A_j B_i B_j^* \\ &+ (A_i A_i^* + 2A_i^* A_i)\, B_i^* B_i A_j^* A_i B_i B_j^* \end{aligned} \tag{135}$$

$$\begin{aligned} &= (A_i^* A_i A_i^* + 2A_i^* A_i A_i^*)\, A_l B_l B_i^* B_i B_i^* \\ &+ \left(A_i^* A_i A_j^* + 2A_j^* A_i A_i^*\right) A_i B_i B_i^* B_j B_i^* \\ &+ (A_i^* A_k A_i^* + 2A_i^* A_k A_i^*)\, A_i B_i B_k^* B_i B_i^* \\ &+ \left(A_i^* A_j A_j^* + 2A_j^* A_j A_i^*\right) A_j B_j B_j^* B_j B_i^* \end{aligned} \tag{136}$$

The multi-step reversal is easily done, and we even reverse our label changes to make it official. The order and placement of all the quaternion factors is restored to the original composition, and the only change is that the prime marks $'$ are removed and B-factor order is reversed. The magic is largely accomplished by virtue of those norm scalars $|B_i|^2 = B_i^* B_i$, which allow us to put these B-factor pairs virtually anywhere in the multi-factor products to restore the order. For D_2, we have the option to just remove the left hand $'$ marks and write, $B_i^* B_j B_k^* B_l$, or to incorporate the reversal, as in, $B_i'^* B_j' B_k'^* B_l' \rightarrow (B_l B_k^* B_j B_i^*)' \rightarrow B_l B_k^* B_j B_i^*$; the ultimate result of the summation is the same, since, like D_1, this re-arrangement symmetry is a property of this partial sum. Now, let's look at D_3.

$$\begin{aligned} D_3 &= (A_i^* A_k A_i^* + 2A_i^* A_k A_i^*)\, A_k B_i'^* B_i' B_k'^* B_k' \\ &+ \left(A_i^* A_j A_j^* + 2A_j^* A_j A_i^*\right) A_i B_i'^* B_j' B_j'^* B_i' \\ &+ \left(A_i^* A_i A_j^* + 2A_j^* A_i A_i^*\right) A_j B_i'^* B_j' B_i'^* B_j' \end{aligned} \tag{137}$$

$$\begin{aligned} &= 3A_i^* A_j A_i^* A_j B_i'^* B_i' B_j'^* B_j' \\ &+ 3A_i^* A_j A_j^* A_i B_i'^* B_i' B_j' B_j'^* \\ &+ 3A_j^* A_i A_i^* A_j B_i'^* B_j' B_i'^* B_j' \end{aligned} \tag{138}$$

$$\begin{aligned} &= 3A_i^* A_j (A_i^* A_j + A_j^* A_i) B_i'^* B_i' B_j'^* B_j' \\ &+ 3(A_i^* A_j)^* (A_i^* A_j)(B_i'^* B_j')^2 \end{aligned} \tag{139}$$

Re-labeling k to j in the first row, juggling factors around, and finally combining the first two rows into one, we end up with two terms to consider (139). Now we use the fact that these terms are summed over all values, $i, j = 1, 2, \cdots, n,\ i \neq j;$, so that for every term, $i = r, j = s$, there's a complementary term, $i = s, j = r$, in the same sum that can be used to pair up terms into scalars using the usual quadratic scalar; $3A_r^* A_s (A_r^* A_s + A_s^* A_r)|B_r|^2 |B_s|^2 + 3A_s^* A_r (A_s^* A_r + A_r^* A_s)|B_s|^2 |B_r|^2 = 3(A_r^* A_s + A_s^* A_r)(A_s^* A_r + A_r^* A_s)|B_s|^2 |B_r|^2$. The sum of these terms, therefore, results in a scalar. Looking at the last term, applying the same technique, we again pair up complementary terms and write, $3|A_r|^2 |A_s|^2 (B_r'^* B_s')^2 + 3|A_s|^2 |A_r|^2 (B_s'^* B_r')^2 = 3|A_r|^2 |A_s|^2 ((B_r'^* B_s')^2 + (B_s'^* B_r')^2)$. There are a few different ways to treat this, but our prefered method is to write, $(B_r'^* B_s')^2 + (B_s'^* B_r')^2 = (B_r'^* B_s' + B_s'^* B_r')^2 - 2|B_r'|^2 |B_s'|^2 = ((B_s B_r^*)' + (B_r B_s^*)')^2 - 2|B_r|^2 |B_s|^2 = (B_s B_r^* + B_r B_s^*)^2 - 2|B_r|^2 |B_s|^2$, removing the $'$ marks as soon as we recognise that the two given elementary rules can be applied. Hence, the third set of terms also evaluates to scalar, and $D_3 \in \mathbb{R}$. Reversing the steps with our right hand B replacements, we return to the same initial expression construction, without the prime marks and with (or optionally without) reversed B-factors;

$$\begin{aligned} D_3 &= (A_i^* A_k A_i^* + 2A_i^* A_k A_i^*)\, A_k B_k B_k^* B_i B_i^* \\ &+ \left(A_i^* A_j A_j^* + 2A_j^* A_j A_i^*\right) A_i B_i B_j^* B_j B_i^* \\ &+ \left(A_i^* A_i A_j^* + 2A_j^* A_i A_i^*\right) A_j B_j B_i^* B_j B_i^* \end{aligned} \tag{140}$$

There are six Gilgamesh bi-quartic terms in the fourth set, D_4, and one in the fifth set, D_5, and these two partial sums can also each independently be shown to be scalar, producing the same results when the prime marks are removed with B-factors reversed. However, we have not yet found a simple way to do this, using general index arguments like those above, and it seems necessary to write out all the terms and collect them together into scalar terms, in order to demonstate this result. This is a long and tedious calculation, which has been performed by symbolic computation only, so far, but not by hand calculation; the results for D_4 and D_5 appear in the 4-term solution given on the next page. The dedicated reader is therefore invited to verify these two partial sum results on his own. For D_4, all six terms can be re-labeled to use the same three ijk indices, and one only needs to evaluate the expression for one unique triple, e.g. $ijk = 123$, to establish the results. For D_5, one only needs to write out the terms for one quartet of unique indicies, so setting $ijkl = 1234$, and evaluating the expression, is sufficient to demonstrate the scalar. There are 24 permutations of the 1234, leading to 24 Gilgamesh type bi-quartic terms to expand and rearrange. For the removal of the prime $'$ marks alone, it is not necessary to reduce the entire expression block for one unique quartet to scalar; some terms can be left in the non-scalar form, e.g. $A_1^* A_2 A_3^* A_4 (B_1'^* B_2' B_3'^* B_4' + B_4'^* B_3' B_2'^* B_1')$, where only the B-factor parts are combined into scalar forms. But, in the case of D_5, we *must reverse* the order of the B-factors. The D_5 only contributes terms when $n \geq 4$; and D_4, only when $n \geq 3$. The determinant can therefore be written,

$$det(H) = \frac{1}{3} \cdot \sum \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) A_l B_i'^* B_j' B_k'^* B_l' \tag{117}$$

$$= \frac{1}{3} \cdot \sum \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) A_l (B_l B_k^* B_j B_i^*)' \tag{141}$$

$$= \frac{1}{3} \cdot \sum \left(A_i^* A_k A_j^* + 2A_j^* A_k A_i^*\right) A_l B_l B_k^* B_j B_i^* \ , \qquad A_u, B_v \in \mathbb{H}_R,\ u, v, \in \{i, j, k, l\} \tag{142}$$

The "Gilgamesh Formula" now allows us to easily complete the solution for the "four term" problem which we attempted above, and we can finally invert the h form of (91), to produce that solution. The result is given below.

Hence, for the " FOUR TERM " linear problem, $(n = 4)$,

$$A_1 q B_1 + A_2 q B_2 + \cdots + A_n q B_n \; = \; C \tag{4}$$

the solution is,

$$\hat{q} \;=\; \frac{\left(\begin{array}{l}
(\,+|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 - |A_4|^2|B_4|^2\,)A_1^* B_1'^* \quad + \\
(\,-|A_1|^2|B_1|^2 + |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 - |A_4|^2|B_4|^2\,)A_2^* B_2'^* \quad + \\
(\,-|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 + |A_3|^2|B_3|^2 - |A_4|^2|B_4|^2\,)A_3^* B_3'^* \quad + \\
(\,-|A_1|^2|B_1|^2 - |A_2|^2|B_2|^2 - |A_3|^2|B_3|^2 + |A_4|^2|B_4|^2\,)A_4^* B_4'^* \\
\bullet \\
+(|B_1|^2 2\alpha_{12} + |A_2|^2 2\beta_{12})A_1^* B_2'^* \;+\; (|B_2|^2 2\alpha_{12} + |A_1|^2 2\beta_{12})A_2^* B_1'^* \\
+(|B_1|^2 2\alpha_{13} + |A_3|^2 2\beta_{13})A_1^* B_3'^* \;+\; (|B_3|^2 2\alpha_{13} + |A_1|^2 2\beta_{13})A_3^* B_1'^* \\
+(|B_1|^2 2\alpha_{14} + |A_4|^2 2\beta_{14})A_1^* B_4'^* \;+\; (|B_4|^2 2\alpha_{14} + |A_1|^2 2\beta_{14})A_4^* B_1'^* \\
+(|B_2|^2 2\alpha_{23} + |A_3|^2 2\beta_{23})A_2^* B_3'^* \;+\; (|B_3|^2 2\alpha_{23} + |A_2|^2 2\beta_{23})A_3^* B_2'^* \\
+(|B_2|^2 2\alpha_{24} + |A_4|^2 2\beta_{24})A_2^* B_4'^* \;+\; (|B_4|^2 2\alpha_{24} + |A_2|^2 2\beta_{24})A_4^* B_2'^* \\
+(|B_3|^2 2\alpha_{34} + |A_4|^2 2\beta_{34})A_3^* B_4'^* \;+\; (|B_4|^2 2\alpha_{34} + |A_3|^2 2\beta_{34})A_4^* B_3'^* \\
\bullet \\
+A_1^* A_2 A_3^* B_1'^* B_3' B_2'^* + A_1^* A_3 A_2^* B_3'^* B_2' B_1'^* + A_2^* A_1 A_3^* B_1'^* B_3' B_2'^* \\
+A_2^* A_3 A_1^* B_3'^* B_1' B_2'^* + A_3^* A_1 A_2^* B_1'^* B_2' B_3'^* + A_3^* A_2 A_1^* B_3'^* B_1' B_2'^* \\
\\
+A_1^* A_2 A_4^* B_1'^* B_4' B_2'^* + A_1^* A_4 A_2^* B_4'^* B_2' B_1'^* + A_2^* A_1 A_4^* B_1'^* B_4' B_2'^* \\
+A_2^* A_4 A_1^* B_4'^* B_1' B_2'^* + A_4^* A_1 A_2^* B_1'^* B_2' B_4'^* + A_4^* A_2 A_1^* B_4'^* B_1' B_2'^* \\
\\
+A_1^* A_3 A_4^* B_1'^* B_4' B_3'^* + A_1^* A_4 A_3^* B_4'^* B_3' B_1'^* + A_3^* A_1 A_4^* B_1'^* B_4' B_3'^* \\
+A_3^* A_4 A_1^* B_4'^* B_1' B_3'^* + A_4^* A_1 A_3^* B_1'^* B_3' B_4'^* + A_4^* A_3 A_1^* B_4'^* B_1' B_3'^* \\
\\
+A_2^* A_3 A_4^* B_2'^* B_4' B_3'^* + A_2^* A_4 A_3^* B_4'^* B_3' B_2'^* + A_3^* A_2 A_4^* B_2'^* B_4' B_3'^* \\
+A_3^* A_4 A_2^* B_4'^* B_2' B_3'^* + A_4^* A_2 A_3^* B_2'^* B_3' B_4'^* + A_4^* A_3 A_2^* B_4'^* B_2' B_3'^*
\end{array}\right)}{\left(\begin{array}{ll}
|A_1|^4|B_1|^4 + |A_2|^4|B_2|^4 + |A_3|^4|B_3|^4 + |A_4|^2|B_4|^2 & D_1 \\
\bullet & \\
+4|A_1|^2|B_1|^2(\alpha_{12}\beta_{12} + \alpha_{13}\beta_{13} + \alpha_{14}\beta_{14}) & \\
+4|A_2|^2|B_2|^2(\alpha_{12}\beta_{12} + \alpha_{23}\beta_{23} + \alpha_{24}\beta_{24}) & \\
+4|A_3|^2|B_3|^2(\alpha_{13}\beta_{13} + \alpha_{23}\beta_{23} + \alpha_{34}\beta_{34}) & D_2 \\
+4|A_4|^2|B_4|^2(\alpha_{14}\beta_{14} + \alpha_{24}\beta_{24} + \alpha_{34}\beta_{34}) & \\
\bullet & \\
-2|A_1|^2|A_2|^2|B_1|^2|B_2|^2 - 2|A_1|^2|A_3|^2|B_1|^2|B_3|^2 - 2|A_1|^2|A_4|^2|B_1|^2|B_4|^2 & \\
-2|A_2|^2|A_3|^2|B_2|^2|B_3|^2 - 2|A_2|^2|A_4|^2|B_2|^2|B_4|^2 - 2|A_3|^2|A_4|^2|B_3|^2|B_4|^2 & \\
 & \\
+4\beta_{12}^2|A_1|^2|A_2|^2 + 4\beta_{13}^2|A_1|^2|A_3|^2 + 4\beta_{14}^2|A_1|^2|A_4|^2 & D_3 \\
+4\beta_{23}^2|A_2|^2|A_3|^2 + 4\beta_{24}^2|A_2|^2|A_4|^2 + 4\beta_{34}^2|A_3|^2|A_4|^2 & \\
+4\alpha_{12}^2|B_1|^2|B_2|^2 + 4\alpha_{13}^2|B_1|^2|B_3|^2 + 4\alpha_{14}^2|B_1|^2|B_4|^2 & \\
+4\alpha_{23}^2|B_2|^2|B_3|^2 + 4\alpha_{24}^2|B_2|^2|B_4|^2 + 4\alpha_{34}^2|B_3|^2|B_4|^2 & \\
\bullet & \\
-4|A_1|^2|B_1|^2(\alpha_{23}\beta_{23} + \alpha_{24}\beta_{24} + \alpha_{34}\beta_{34}) & \\
-4|A_2|^2|B_2|^2(\alpha_{13}\beta_{13} + \alpha_{14}\beta_{14} + \alpha_{34}\beta_{34}) & \\
-4|A_3|^2|B_3|^2(\alpha_{12}\beta_{12} + \alpha_{14}\beta_{14} + \alpha_{24}\beta_{24}) & \\
-4|A_4|^2|B_4|^2(\alpha_{12}\beta_{12} + \alpha_{13}\beta_{13} + \alpha_{23}\beta_{23}) & D_4 \\
 & \\
+8|A_1|^2(\alpha_{23}\beta_{12}\beta_{13} + \alpha_{24}\beta_{12}\beta_{14} + \alpha_{34}\beta_{13}\beta_{14}) \;\; +8|B_1|^2(\beta_{23}\alpha_{12}\alpha_{13} + \beta_{24}\alpha_{12}\alpha_{14} + \beta_{34}\alpha_{13}\alpha_{14}) & \\
+8|A_2|^2(\alpha_{13}\beta_{12}\beta_{23} + \alpha_{14}\beta_{12}\beta_{24} + \alpha_{34}\beta_{23}\beta_{24}) \;\; +8|B_2|^2(\beta_{13}\alpha_{12}\alpha_{23} + \beta_{14}\alpha_{12}\alpha_{24} + \beta_{34}\alpha_{23}\alpha_{24}) & \\
+8|A_3|^2(\alpha_{24}\beta_{23}\beta_{34} + \alpha_{12}\beta_{13}\beta_{23} + \alpha_{14}\beta_{13}\beta_{34}) \;\; +8|B_3|^2(\beta_{24}\alpha_{23}\alpha_{34} + \beta_{12}\alpha_{13}\alpha_{23} + \beta_{14}\alpha_{13}\alpha_{34}) & \\
+8|A_4|^2(\alpha_{12}\beta_{14}\beta_{24} + \alpha_{13}\beta_{14}\beta_{34} + \alpha_{23}\beta_{24}\beta_{34}) \;\; +8|B_4|^2(\beta_{12}\alpha_{14}\alpha_{24} + \beta_{13}\alpha_{14}\alpha_{34} + \beta_{23}\alpha_{24}\alpha_{34}) & \\
\bullet & \\
-16\alpha_{12}\alpha_{34}\beta_{12}\beta_{34} - 16\alpha_{13}\alpha_{24}\beta_{13}\beta_{24} - 16\alpha_{14}\alpha_{23}\beta_{14}\beta_{23} & D_5 \\
+16\alpha_{13}\alpha_{24}(\beta_{12}\beta_{34} + \beta_{14}\beta_{23}) + 16\beta_{13}\beta_{24}(\alpha_{12}\alpha_{34} + \alpha_{14}\alpha_{23}) & \\
+8\gamma_{1234}(\beta_{12}\beta_{34} - \beta_{13}\beta_{24} + \beta_{14}\beta_{23}) + 8\delta_{1234}(\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24} + \alpha_{14}\alpha_{23}) - 8\gamma_{1234}\delta_{1234} &
\end{array}\right)} \cdot \hat{C} \tag{143}$$

where,

$$2\alpha_{ij} = A_i^* A_j + A_j^* A_i\,, \qquad 2\gamma_{1234} = A_1^* A_2 A_3^* A_4 + (A_1^* A_2 A_3^* A_4)^* \qquad i,j = 1,2,\cdots,n=4;\; i \neq j$$

$$2\,\beta_{ij} = B_i^* B_j + B_j^* B_i\,, \qquad 2\,\delta_{1234} = B_1^* B_2 B_3^* B_4 + (B_1^* B_2 B_3^* B_4)^*$$

$$A_k, B_k, C, q \in \mathbb{H}_R\,;\;\; B_k' \in \mathbb{H}_L\,;\;\; k = 1,2,\cdots,n=4. \qquad \alpha_{ij}, \beta_{ij}, \gamma_{1234}, \delta_{1234} \in \mathbb{R};$$

This " four term " solution contains all the detail necessary to infer the " irreducible " n-term solution, and so write it down in similar format. Using double subscript notation for the scalars, α_{jk} and β_{jk}, allows us to easily generalise these expressions. Inspection shows there are essentially three different expression blocks in the numerator. In the formula shown in eqn (143), these blocks are the separated by the • bullets. The first block contains all the terms of form $A_i^* B_i^{\prime *}$, which produce $A_i^* C B_i^*$, that have two identical indicies. There are thus $\boldsymbol{n}$ such terms in this block. The second block contains all the $A_i^* B_j^{\prime *}$ terms, which produce $A_i^* C B_j^*$, with a pair of dissimilar indices, $i \neq j$, so there are $\boldsymbol{n(n-1)}$ of them. The third block contains all the bi-cubic terms $A_i^* A_j A_k^* B_a^{\prime *} B_b^{\prime} B_c^{\prime *}$, which produce $A_i^* A_j A_k^* C B_c^* B_b B_a^*$, with all three subscript indicies unique. In a set of $\boldsymbol{n}$ indicies, there are $\boldsymbol{n!/(3!(n-3)!)}$ ways to select the $\boldsymbol{3}$ indicies for such terms. For each such ijk unique triple, there are $\boldsymbol{3! = 6}$ ways to permute, and so 6-terms containing the same three different indicies. This means the third expression block breaks down further into $\boldsymbol{n!/(3!(n-3)!)}$ 6-term sub-blocks. Each of these 6-term blocks can be reduced to a 4-term block, as illustrated in (88a), to achieve a minimal term count. But, in our solutions we keep the 6-term form, because it's the easiest form with which to see the index permutation symmetry by simple inspection. However, the ability to reduce these 6-term blocks means that they really contribute only $\boldsymbol{4 \cdot n!/(3!(n-3)!)}$ irreducible quaternions to the numerator term count. So, the number of irreducible terms in the numerator of the n-term solution is,

NO. OF IRREDUCIBLE QUATERNION TERMS:

$$n + n \cdot (n-1) + \frac{4 \cdot n!}{3! \cdot (n-3)!} \quad = \quad n^2 + \frac{2}{3} \cdot n \cdot (n-1) \cdot (n-2) \quad = \quad (2n^3 - 3n^2 + 4n)/3 \tag{144}$$

The R-H-S of this formula holds for all $\boldsymbol{n \geq 1}$, despite the fact that, on the L-H-S, when $\boldsymbol{n = 1}$, or $\boldsymbol{n = 2}$, the factor $\boldsymbol{(n-3)!}$ is usually undefined, (we assume the convention $\boldsymbol{0! = 1}$, holds, when $\boldsymbol{n = 3}$). Therefore, our original $\boldsymbol{n^3}$ numerator term count, with the primal bi-cubic quaternion terms $A^* A A^* B^* B B^*$, can be reduced by at most $\boldsymbol{n^3} - (2\boldsymbol{n}^3 - 3\boldsymbol{n}^2 + 4\boldsymbol{n})/3 = \boldsymbol{n}(\boldsymbol{n} - 1)(\boldsymbol{n} + 4)/3$ quaternion terms for the general n-term problem. Problems with special A_k, B_k parameters, may, of course, allow for further reductions in term count, but the general problem cannot be simplified any further.

THE N-TERM LINEAR PROBLEM.

The primary solution to the n-term linear equation is given in (A-61), with the method and solution, (103), based on the Gilgamesh quaternion expansions, and the irreducible form of this solution can now also be constructed by extending the expression blocks shown in the 4-term solution given in (143). The further n-term problems, i.e. with $n > 4$, add nothing new to the form of the solutions. There are essentially 3 expression blocks in the numerator and 5 blocks in the denominator. Each expression block has a readily recognisable index pattern that lends to easy extension from 4 to n indicies, as the new terms introduced by the n-term problem follow the forms already present in these exprssion blocks. The 5 partial sums, D_1, D_2, D_3, D_4, D_5, that make up the determinant in the denominator of (143) are all separated by • bullets, and marked on the R-H-S with their corresponding labels, to identify these expression blocks as the scalars that result from these partial sums. The 4-term problem has only one unique quartet, $ijkl = 1234$, which produces the D_5 scalar result appearing at the bottom of this formula. When $n > 4$, there will be several unique quartets, $ijkl = abcd$, each of which will contribute another expression block equal to that shown for the 4-term solution; one simply replaces the 1234 indicies by the $abcd$ for each of the unique quartets and adds the new expression sub-block to this block. In this way, we can write down the solution to the n-term problem, by just inspecting the 4-term solution alone, and the n-term problem can therefore also be considered to be completely solved.

Further simplification of the denominator is available by combining expression block D_2 with the first sub-block shown in D_4, and other re-arrangements are also possible. However, we prefer to leave the formula this way, with the partial sums clearly separated, because it renders the solution more intelligible. As previously discussed, the 6-term blocks of unique ijk triples in the numerator can also be reduced, by replacing with the equivalent 4-term block, illustrated in (88a), for minimal term count, or replacing with the equivalent 9-term block, illustrated in (88b), for maximum reduction in the "order" of terms present. This all means that our formula (143) can be reduced further, and is "a reduced" but not the ultimate "irreducible" formula for the solution. This "reduced" formula is just more convienent to remember the form of the expressions and to enable extension to the n-term solutions; and the "irreducible" form can always be readily deduced from it.

Apart from the many quadratic scalars, α_{ij} and β_{ij}, the denominators with D_5 terms present contain a number of " quartic scalars, " $\gamma_{abcd} = S(A_a^* A_b A_c^* A_d)$ and $\delta_{abcd} = S(B_a^* B_b B_c^* B_d)$, preventing hand transform without factor reversal.

II. LINEAR EQUATIONS IN TWO VARIABLES.

$$
\begin{aligned}
A_1\, p\, B_1 \quad &+ \quad C_1\, q\, D_1 \quad = \quad E_1 \\[2ex]
A_2\, p\, B_2 \quad &+ \quad C_2\, q\, D_2 \quad = \quad E_2
\end{aligned}
\tag{145}
$$

$$
\begin{aligned}
A_{11}\, p\, B_{11} + A_{12}\, p\, B_{12} + \cdots + A_{1m_1}\, p\, B_{1m_1} \quad &+ \quad C_{11}\, q\, D_{11} + C_{12}\, q\, D_{12} + \cdots + C_{1n_1}\, q\, D_{1n_1} \quad = \quad E_1 \\[2ex]
A_{21}\, p\, B_{21} + A_{22}\, p\, B_{22} + \cdots + A_{2m_2}\, p\, B_{2m_2} \quad &+ \quad C_{21}\, q\, D_{21} + C_{22}\, q\, D_{22} + \cdots + C_{2n_2}\, q\, D_{2n_2} \quad = \quad E_2
\end{aligned}
\tag{146}
$$

We call the linear system "square" when each independent variable contributes no more than one term in each equation, (145), and we call the linear system "rectangular" otherwise, (146). The one hand quaternion rectangular system can easily be converted into a two-hand algebra square, and then solved readily. Consider the square (145);

$$
\begin{array}{rcccl}
A_1 B_1' \hat{p} & + & C_1 D_1' \hat{q} & = & \hat{E}_1 \\
A_2 B_2' \hat{p} & + & C_2 D_2' \hat{q} & = & \hat{E}_2
\end{array}
\quad \Longrightarrow \quad
\begin{pmatrix} A_1 B_1' & C_1 D_1' \\ A_2 B_2' & C_2 D_2' \end{pmatrix}
\begin{pmatrix} \hat{p} \\ \hat{q} \end{pmatrix}
=
\begin{pmatrix} \hat{E}_1 \\ \hat{E}_2 \end{pmatrix}
\tag{147}
$$

Multiplying the top equation by, $|C_2 D_2'|^2 (C_1 D_1')^*$, and the bottom equation by, $|C_1 D_1'|^2 (C_2 D_2')^*$, then subtracting equations to isolate the independent p variable, and working similarily for q, we get,

$$
\begin{aligned}
(|C_2 D_2'|^2 (C_1 D_1')^* A_1 B_1' - |C_1 D_1'|^2 (C_2 D_2')^* A_2 B_2')\, \hat{p} \quad &= \quad |C_2 D_2'|^2 (C_1 D_1')^* \hat{E}_1 - |C_1 D_1'|^2 (C_2 D_2')^* \hat{E}_2 \\
(|A_2 B_2'|^2 (A_1 B_1')^* C_1 D_1' - |A_1 B_1'|^2 (A_2 B_2')^* C_2 D_2')\, \hat{q} \quad &= \quad |A_2 B_2'|^2 (A_1 B_1')^* \hat{E}_1 - |A_1 B_1'|^2 (A_2 B_2')^* \hat{E}_2
\end{aligned}
\tag{148}
$$

This is equivalent to multiplying both sides of the matrix equation by,

$$
\begin{pmatrix}
|C_2 D_2'|^2 (C_1 D_1')^* & -|C_1 D_1'|^2 (C_2 D_2')^* \\
|A_2 B_2'|^2 (A_1 B_1')^* & -|A_1 B_1'|^2 (A_2 B_2')^*
\end{pmatrix}
\tag{149}
$$

so that we can then write the solution,

$$
\begin{pmatrix} \hat{p} \\ \\ \hat{q} \end{pmatrix}
=
\begin{pmatrix} \frac{1}{d_1} & 0 \\ 0 & \frac{1}{d_2} \end{pmatrix}
\cdot
\begin{pmatrix}
|C_2 D_2'|^2 (C_1 D_1')^* & -|C_1 D_1'|^2 (C_2 D_2')^* \\ \\
|A_2 B_2'|^2 (A_1 B_1')^* & -|A_1 B_1'|^2 (A_2 B_2')^*
\end{pmatrix}
\cdot
\begin{pmatrix} \hat{E}_1 \\ \\ \hat{E}_2 \end{pmatrix}
\tag{150}
$$

where,

$$
\begin{aligned}
d_1 &= (|C_2 D_2'|^2 (C_1 D_1')^* A_1 B_1' - |C_1 D_1'|^2 (C_2 D_2')^* A_2 B_2') = |C_1|^2 |C_2|^2 |D_1'|^2 |D_2'|^2 (C_1 \backslash A_1 \cdot D_1' \backslash B_1' - C_2 \backslash A_2 \cdot D_2' \backslash B_2') \\
d_2 &= (|A_2 B_2'|^2 (A_1 B_1')^* C_1 D_1' - |A_1 B_1'|^2 (A_2 B_2')^* C_2 D_2') = |A_1|^2 |A_2|^2 |B_1'|^2 |B_2'|^2 (A_1 \backslash C_1 \cdot B_1' \backslash D_1' - A_2 \backslash C_2 \cdot B_2' \backslash D_2')
\end{aligned}
$$

The d_1 and d_2 are 2-term bilinear hexpe factors[9], and their inverses are therefore given by eqn (53); substituting, $A_1 \to C_1 \backslash A_1$, $B_1 \to D_1 / B_1$, and $A_2 \to C_2 \backslash A_2$, $B_2 \to D_2 / B_2$, for the factors in eqn (53), the inverse of d_1 can be written;

$$
d_1^{-1} \quad = \quad \left(\frac{
\begin{array}{l}
\left(|C_1 \backslash A_1|^2 \cdot |D_1/B_1|^2 - |C_2 \backslash A_2|^2 \cdot |D_2/B_2|^2 \right) \cdot ((C_1 \backslash A_1)^* \cdot (D_1/B_1)'^* + (C_2 \backslash A_2)^* \cdot (D_2/B_2)'^*) \\
+2 \cdot (|C_2 \backslash A_2|^2 \cdot b + |D_1/B_1|^2 \cdot a) \cdot (C_1 \backslash A_1)^* \cdot (D_2/B_2)'^* - 2 \cdot (|C_1 \backslash A_1|^2 \cdot b + |D_2/B_2|^2 \cdot a) \cdot (C_2 \backslash A_2)^* \cdot (D_1/B_1)'^*
\end{array}
}{
\begin{array}{l}
|C_1|^2 |C_2|^2 |D_1|^2 |D_2|^2 \left((|C_1 \backslash A_1|^2 \cdot |D_1/B_1|^2 - |C_2 \backslash A_2|^2 \cdot |D_2/B_2|^2)^2 \right. \\
\left. +4 \cdot (|C_1 \backslash A_1|^2 \cdot |D_1/B_1|^2 + |C_2 \backslash A_2|^2 \cdot |D_2/B_2|^2) \cdot ab + 4 \cdot |D_1/B_1|^2 \cdot |D_2/B_2|^2 \cdot a^2 + 4 \cdot |C_1 \backslash A_1|^2 \cdot |C_2 \backslash A_2|^2 \cdot b^2 \right)
\end{array}
} \right)
\tag{151}
$$

where, $2a = -(C_1 \backslash A_1)^* \cdot (C_2 \backslash A_2) - ((C_1 \backslash A_1)^* \cdot (C_2 \backslash A_2))^*$, $2b = (D_1/B_1)^* \cdot (D_2/B_2) + ((D_1/B_1)^* \cdot (D_2/B_2))^*$

and with similar substitutions, the inverse of d_2 can be written;

$$
d_2^{-1} \quad = \quad \left(\frac{
\begin{array}{l}
\left(|A_1 \backslash C_1|^2 \cdot |B_1/D_1|^2 - |A_2 \backslash C_2|^2 \cdot |B_2/D_2|^2 \right) \cdot ((A_1 \backslash C_1)^* \cdot (B_1/D_1)'^* + (A_2 \backslash C_2)^* \cdot (B_2/D_2)'^*) \\
+2 \cdot (|A_2 \backslash C_2|^2 \cdot b + |B_1/D_1|^2 \cdot a) \cdot (A_1 \backslash C_1)^* \cdot (B_2/D_2)'^* - 2 \cdot (|A_1 \backslash C_1|^2 \cdot b + |B_2/D_2|^2 \cdot a) \cdot (A_2 \backslash C_2)^* \cdot (B_1/D_1)'^*
\end{array}
}{
\begin{array}{l}
|A_1|^2 |A_2|^2 |B_1|^2 |B_2|^2 \left((|A_1 \backslash C_1|^2 \cdot |B_1/D_1|^2 - |A_2 \backslash C_2|^2 \cdot |B_2/D_2|^2)^2 \right. \\
\left. +4 \cdot (|A_1 \backslash C_1|^2 \cdot |B_1/D_1|^2 + |A_2 \backslash C_2|^2 \cdot |B_2/D_2|^2) \cdot ab + 4 \cdot |B_1/D_1|^2 \cdot |B_2/D_2|^2 \cdot a^2 + 4 \cdot |A_1 \backslash C_1|^2 \cdot |A_2 \backslash C_2|^2 \cdot b^2 \right)
\end{array}
} \right)
\tag{152}
$$

where, $2a = -(A_1 \backslash C_1)^* \cdot (A_2 \backslash C_2) - ((A_1 \backslash C_1)^* \cdot (A_2 \backslash C_2))^*$, $2b = (B_1/D_1)^* \cdot (B_2/D_2) + ((B_1/D_1)^* \cdot (B_2/D_2))^*$

This puts all the non-scalar parameters into the numerator, and the denominators then contain only scalar terms; eqn (150) can then be resolved into the one-hand quaternion form by moving the usual left handed quaternions over to the R-H-S of the $\hat{E}$ factors. The final solution, (p, q), to the simultaneous equation system (145) is then,

$$p = \left(\frac{\begin{array}{c} +(|C_1\backslash A_1|^2 \cdot |D_1/B_1|^2 - |C_2\backslash A_2|^2 \cdot |D_2/B_2|^2) \cdot (C_1\backslash A_1)^* \cdot ((C_1\backslash E_1)/D_1 - (C_2\backslash E_2)/D_2) \cdot (D_1/B_1)^* \\ +(|C_1\backslash A_1|^2 \cdot |D_1/B_1|^2 - |C_2\backslash A_2|^2 \cdot |D_2/B_2|^2) \cdot (C_2\backslash A_2)^* \cdot ((C_1\backslash E_1)/D_1 - (C_2\backslash E_2)/D_2) \cdot (D_2/B_2)^* \\ +2 \cdot (|C_2\backslash A_2|^2 \cdot b + |D_1/B_1|^2 \cdot a) \cdot (C_1\backslash A_1)^* \cdot ((C_1\backslash E_1)/D_1 - (C_2\backslash E_2)/D_2) \cdot (D_2/B_2)^* \\ -2 \cdot (|C_1\backslash A_1|^2 \cdot b + |D_2/B_2|^2 \cdot a) \cdot (C_2\backslash A_2)^* \cdot ((C_1\backslash E_1)/D_1 - (C_2\backslash E_2)/D_2) \cdot (D_1/B_1)^* \end{array}}{\begin{array}{l} ((|C_1\backslash A_1|^2 \cdot |D_1/B_1|^2 - |C_2\backslash A_2|^2 \cdot |D_2/B_2|^2)^2 \\ +4 \cdot (|C_1\backslash A_1|^2 \cdot |D_1/B_1|^2 + |C_2\backslash A_2|^2 \cdot |D_2/B_2|^2) \cdot ab + 4 \cdot |D_1/B_1|^2 \cdot |D_2/B_2|^2 \cdot a^2 + 4 \cdot |C_1\backslash A_1|^2 \cdot |C_2\backslash A_2|^2 \cdot b^2) \end{array}} \right) \tag{153}$$

where, $2a = -(C_1\backslash A_1)^* \cdot (C_2\backslash A_2) - ((C_1\backslash A_1)^* \cdot (C_2\backslash A_2))^*, \;\; 2b = (D_1/B_1)^* \cdot (D_2/B_2) + ((D_1/B_1)^* \cdot (D_2/B_2))^*$

and,

$$q = \left(\frac{\begin{array}{c} +(|A_1\backslash C_1|^2 \cdot |B_1/D_1|^2 - |A_2\backslash C_2|^2 \cdot |B_2/D_2|^2) \cdot (A_1\backslash C_1)^* \cdot ((A_1\backslash E_1)/B_1 - (A_2\backslash E_2)/B_2) \cdot (B_1/D_1)^* \\ +(|A_1\backslash C_1|^2 \cdot |B_1/D_1|^2 - |A_2\backslash C_2|^2 \cdot |B_2/D_2|^2) \cdot (A_2\backslash C_2)^* \cdot ((A_1\backslash E_1)/B_1 - (A_2\backslash E_2)/B_2) \cdot (B_2/D_2)^* \\ +2 \cdot (|A_2\backslash C_2|^2 \cdot b + |B_1/D_1|^2 \cdot a) \cdot (A_1\backslash C_1)^* \cdot ((A_1\backslash E_1)/B_1 - (A_2\backslash E_2)/B_2) \cdot (B_2/D_2)^* \\ -2 \cdot (|A_1\backslash C_1|^2 \cdot b + |B_2/D_2|^2 \cdot a) \cdot (A_2\backslash C_2)^* \cdot ((A_1\backslash E_1)/B_1 - (A_2\backslash E_2)/B_2) \cdot (B_1/D_1)^* \end{array}}{\begin{array}{l} ((|A_1\backslash C_1|^2 \cdot |B_1/D_1|^2 - |A_2\backslash C_2|^2 \cdot |B_2/D_2|^2)^2 \\ +4 \cdot (|A_1\backslash C_1|^2 \cdot |B_1/D_1|^2 + |A_2\backslash C_2|^2 \cdot |B_2/D_2|^2) \cdot ab + 4 \cdot |B_1/D_1|^2 \cdot |B_2/D_2|^2 \cdot a^2 + 4 \cdot |A_1\backslash C_1|^2 \cdot |A_2\backslash C_2|^2 \cdot b^2) \end{array}} \right) \tag{154}$$

where, $2a = -(A_1\backslash C_1)^* \cdot (A_2\backslash C_2) - ((A_1\backslash C_1)^* \cdot (A_2\backslash C_2))^*, \;\; 2b = (B_1/D_1)^* \cdot (B_2/D_2) + ((B_1/D_1)^* \cdot (B_2/D_2))^*$

The solution to the linear system exists when the scalar denominators are of non-vanishing values. The second, "rectangular" system (146), transforms into "square" two-hand;

$$\begin{aligned} \ddot{h}(A_1, B_1)_{m_1} \cdot \hat{p} \;+\; \ddot{h}(C_1, D_1)_{n_1} \cdot \hat{q} &= \hat{E}_1 \\ \ddot{h}(A_2, B_2)_{m_2} \cdot \hat{p} \;+\; \ddot{h}(C_2, D_2)_{n_2} \cdot \hat{q} &= \hat{E}_2 \end{aligned} \tag{155}$$

where, this notation, $\ddot{h}(A_j, B_j)_m$, adds the second subscripted index, $A_{jk}B'_{jk}, \; k = 1, 2, \ldots, m$; and the appropriate hand transform $'$ marks, to form the usual bilinear hexpe expressions;

$$\begin{aligned} \ddot{h}(A_1, B_1)_{m_1} &= A_{11}B'_{11} + A_{12}B'_{12} + \cdots A_{1m_1}B'_{1m_1} \\ \ddot{h}(A_2, B_2)_{m_2} &= A_{21}B'_{21} + A_{22}B'_{22} + \cdots A_{2m_2}B'_{2m_2} \\ \ddot{h}(C_1, D_1)_{n_1} &= C_{11}D'_{11} + C_{12}D'_{12} + \cdots C_{1n_1}D'_{1n_1} \\ \ddot{h}(C_2, D_2)_{n_2} &= C_{11}D'_{11} + C_{12}D'_{12} + \cdots C_{1n_2}D'_{1n_2} \end{aligned} \tag{156}$$

To solve this system, we may choose to multiply by the inverse, $\ddot{h}_m^{-1}$, or the adjoint, $\ddot{h}_m^{\dagger}$, both of which we now know how to construct, and use these to reduce the factors to scalars; then, eliminate one or the other independent variable to find the solution. When the four determinant values are non-zero, i.e. $\det(\ddot{h}(A_1, B_1)_{m_1}) \neq 0$, $\det(\ddot{h}(A_2, B_2)_{m_2}) \neq 0$, $\det(\ddot{h}(C_1, D_1)_{n_1}) \neq 0$, $\det(\ddot{h}(C_2, D_2)_{m_1}) \neq 0$, we may transform this system into;

$$(\ddot{h}(C_1, D_1)_{n_1}^{-1}\ddot{h}(A_1, B_1)_{m_1} - \ddot{h}(C_2, D_2)_{n_2}^{-1}\ddot{h}(A_2, B_2)_{m_2}) \cdot \hat{p} = \ddot{h}(C_1, D_1)_{n_1}^{-1}\hat{E}_1 - \ddot{h}(C_2, D_2)_{n_2}^{-1}\hat{E}_2 \tag{157}$$

$$(\ddot{h}(A_1, B_1)_{m_1}^{-1}\ddot{h}(C_1, D_1)_{n_1} - \ddot{h}(A_2, B_2)_{m_2}^{-1}\ddot{h}(C_2, D_2)_{n_2}) \cdot \hat{q} = \ddot{h}(A_1, B_1)_{m_1}^{-1}\hat{E}_1 - \ddot{h}(A_2, B_2)_{m_2}^{-1}\hat{E}_2 \tag{158}$$

and re-write these equations,

$$\hat{p} \;=\; \frac{\ddot{h}(C_1,D_1)^{-1}_{n_1}\hat{E}_1 - \ddot{h}(C_2,D_2)^{-1}_{n_2}\hat{E}_2}{\vdash (\ddot{h}(C_1,D_1)^{-1}_{n_1}\ddot{h}(A_1,B_1)_{m_1} - \ddot{h}(C_2,D_2)^{-1}_{n_2}\ddot{h}(A_2,B_2)_{m_2})} \tag{159}$$

$$\hat{q} \;=\; \frac{\ddot{h}(A_1,B_1)^{-1}_{m_1}\hat{E}_1 - \ddot{h}(A_2,B_2)^{-1}_{m_2}\hat{E}_2}{\vdash (\ddot{h}(A_1,B_1)^{-1}_{m_1}\ddot{h}(C_1,D_1)_{n_1} - \ddot{h}(A_2,B_2)^{-1}_{m_2}\ddot{h}(C_2,D_2)_{n_2})} \tag{160}$$

The denominators in these expressions for (p,q) are not scalars, but general hexpe numbers; hence, the use of the usual $\vdash$ in the denominator, to indicate division from the left in fractional notation. When these hexpe numbers in the denominator are invertible, we can obtain the solution to the rectangular system.

Now, $\ddot{h}(A_1,B_1)_{m_1}$, is an m_1-term bilinear expression, and its inverse, $\ddot{h}(A_1,B_1)^{-1}_{m_1}$, according to our established formula (144), can be reduced to have a minimum term count of $(2m_1^3 - 3m_1^2 + 4m_1)/3$ irreducible quaternion terms; if this is greater than 16, then the count can be reduced further, but only by breaking the quaternions up into their components. This means that the product, $\ddot{h}(A_1,B_1)^{-1}_{m_1}\ddot{h}(C_1,D_1)_{n_1}$, will have at least $n_1\cdot(2m_1^3 - 3m_1^2 + 4m_1)/3$ quaternion terms in its bilinear form. So, the denominator in the formula for $\hat{q}$ is an n-term bilinear hexpe number, $\ddot{h}_n$, where $n = n_1\cdot(2m_1^3 - 3m_1^2 + 4m_1)/3 + n_2\cdot(2m_2^3 - 3m_2^2 + 4m_2)/3$. Similarly, the denominator in the formula for $\hat{p}$ is an n-term bilinear, with $n = m_1\cdot(2n_1^3 - 3n_1^2 + 4n_1)/3 + m_2\cdot(2n_2^3 - 3n_2^2 + 4n_2)/3$.

When these n-term bilinear inverses, $\ddot{h}_n^{-1}$, in the denominators for (p,q), do not exist, there may be special case solutions, as in the case in the linear equations in one variable. This method of approach can be extended to find solutions to the general linear system of equations in N variables.

Applications and Examples.

In [YT1], Tian[2] discusses three equations $ax - xb = c$, $ax - x^*b = c$, and $x^*ax = b$ in quaternions. We solve the first two to illustrate the two-hand approach; the third is quadratic, so beyond the scope of this paper.

EXAMPLE 1 : $ax - xb = c$ $\qquad a,b,c,x \in \mathbb{H}_R$

$$\begin{aligned}
&ax - xb = c \quad\rightarrow\quad a\hat{x} - b'\hat{x} = \hat{c} \quad\rightarrow\quad (a - b')\hat{x} = \hat{c}\\
&(a-b')^{*R}(a-b')\hat{x} = (a-b')^{*R}\hat{c} \quad\rightarrow\quad (a^* - b')(a-b')\hat{x} = (a^* - b')\hat{c}\\
&(|a|^2 - (a+a^*)b' + b'^2)\hat{x} = (a^* - b')\hat{c}\\
&(|a|^2 - (a+a^*)b' + b'^2)^{*L}(|a|^2 - (a+a^*)b' + b'^2)\hat{x} = (|a|^2 - (a+a^*)b' + b'^2)^{*L}(a^* - b')\hat{c}\\
&(|a|^2 - (a+a^*)b'^* + (b'^*)^2)(|a|^2 - (a+a^*)b' + b'^2)\hat{x} = (|a|^2 - (a+a^*)b'^* + (b'^*)^2)(a^* - b')\hat{c}
\end{aligned} \tag{161}$$

Simplifying the factors and rearranging,

$$\hat{x} \;=\; \left(\frac{|b|^2a - |a|^2b' + |a|^2a^* - |b|^2b'^* - ((a+a^*) - (b+b^*))a^*b'^*}{(|a|^2 - |b|^2)^2 - (|a|^2 + |b|^2)(a+a^*)(b+b^*) + |b|^2(a+a^*)^2 + |a|^2(b+b^*)^2}\right)\hat{c} \tag{162}$$

$$\therefore \qquad x \;=\; \frac{|b|^2ac - |a|^2cb + |a|^2a^*c - |b|^2cb^* - ((a+a^*) - (b+b^*))a^*cb^*}{(|a|^2 - |b|^2)^2 - (|a|^2 + |b|^2)(a+a^*)(b+b^*) + |b|^2(a+a^*)^2 + |a|^2(b+b^*)^2} \tag{163}$$

This result could, of course, be obtained more directly by using the eqn (54) formula. But, it is always useful to see the same problem solved different ways, especially since slightly different expressions result from the various alterations in method, and it often takes some effort to show that the variations are actually the same[10]. In fact, we previously solved this very equation in our "Quatro-Quaternion" paper [PJ3], the solution being reproduced in this paper above in eqns (QQ-72) and (QQ-73). But, in [PJ3] we used the results from a rather lengthy method to arrive at this general unique solution; we re-solve this above in eqns (51) and (52) in the developed method of this paper. Finally, here again, we exploit the fact that the 2-term problem can be solved by building up either of the conjugated cubic factors, $(h^{*R}h)^{*L}h^{*R}$ or $(h^{*L}h)^{*R}h^{*L}$, on the L-H-S of the equation, to reduce it to scalar, to demonstrate this simpler approach for Tian's statement of the same problem. This method is special, however, and does not generalize to the n-term; unlike the Gilgamesh Solution which is general and solves this and the n-term also.

EXAMPLE 2 : $ax - x^*b = c$ $\qquad a, b, c, x \in \mathbb{H}_R$

$$\begin{array}{rcl} a\,x \;-\; x^*\,b &=& c \\ -b^*x \;+\; x^*a^* &=& c^* \end{array} \;\rightarrow\; \begin{array}{rcl} a\,\hat{x} \;-\; b'\,\hat{x}^* &=& \hat{c} \\ -b^*\hat{x} \;+\; a'^*\hat{x}^* &=& \hat{c}^* \end{array} \;\rightarrow\; \begin{array}{rcl} |a|^2b'^*a\,\hat{x} \;-\; |a|^2b'^*b'\,\hat{x}^* &=& |a|^2b'^*\hat{c} \\ -|b|^2a'b^*\hat{x} \;+\; |b|^2a'a'^*\hat{x}^* &=& |b|^2a'\hat{c}^* \end{array} \tag{164}$$

We treat this as a problem in two variables (x , x^*), and conjugate the given equation to obtain the second equation, in order to form a pair of simultaneous equations in the two variables. Then, we solve the linear system.

$$(|a|^2ab'^* - |b|^2b^*a')\hat{x} = (|a|^2b'^*\hat{c} + |b|^2a'\hat{c}^*) \tag{165}$$

$$(|a|^2ab'^* - |b|^2b^*a')^{*R}(|a|^2ab'^* - |b|^2b^*a')\hat{x} = (|a|^2ab'^* - |b|^2b^*a')^{*R}(|a|^2b'^*\hat{c} + |b|^2a'\hat{c}^*) \tag{166}$$

$$(|a|^2a^*b'^* - |b|^2ba')(|a|^2ab'^* - |b|^2b^*a')\hat{x} = (|a|^2a^*b'^* - |b|^2ba')(|a|^2b'^*\hat{c} + |b|^2a'\hat{c}^*) \tag{167}$$

$$(+|a|^6b'^*b'^* - |a|^2|b|^2a^*b^*b'^*a' - |a|^2|b|^2baa'b'^* + |b|^6a'a')\hat{x} = (|a|^2a^*b'^* - |b|^2ba')(|a|^2b'^*\hat{c} + |b|^2a'\hat{c}^*) \tag{168}$$

multiplying both sides by, $(|a|^6b'^*b'^* - |a|^2|b|^2a^*b^*b'^*a' - |a|^2|b|^2baa'b'^* + |b|^6a'a')^{*L}$, symplifying and rearranging,

$$\hat{x} = \left(\frac{\begin{array}{l} |a|^6a^*b' - |a|^4bb'b'a' - |a|^2|b|^2a^*b^*a^*a'^* \\ +|b|^4a^*a'^*b'a' - |a|^4bb'a'^*b'^* + |a|^2|b|^2babb' \\ +|b|^4a^*a'^*a'^*b'^* - |b|^6ba'^* \end{array}}{|a|^2|b|^2(|a|^2 - |b|^2)^2((|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2)} \right) (|a|^2b'^*\hat{c} + |b|^2a'\hat{c}^*) \tag{169}$$

$$\therefore \qquad x = \left(\frac{\begin{array}{l} |a|^6|b|^2a^*(\;|a|^2c + c^*ab\;) - |a|^4b(\;|a|^2cb^* + |b|^2c^*a\;)abb - |a|^4|b|^2a^*b^*a^*(\;cb^*a^* + |b|^2c^*\;) \\ +|b|^4a^*(\;|a|^2cb^* + |b|^2c^*a\;)aba^* - |a|^4b(\;|a|^2cb^* + |b|^2c^*a\;)b^*a^*b + |a|^2|b|^4bab(\;|a|^2c + c^*ab\;) \\ +|b|^4a^*(\;|a|^2cb^* + |b|^2c^*a\;)b^*a^*a^* - |b|^6|a|^2b(\;cb^*a^* + |b|^2c^*\;) \end{array}}{|a|^2|b|^2(|a|^2 - |b|^2)^2(\;(|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2\;)} \right) \tag{170}$$

With some more re-arranging this somewhat raw formula is further reduced to,

$$\therefore \qquad x = \left(\frac{\begin{array}{l} (|a|^2 + |b|^2)(\;|a|^2a^*c + a^*c^*ab + bcb^*a^* + |b|^2bc^*\;) \\ -((ab) + (ab)^*)(\;|b|^2a^*c^* + bc^*ab + a^*cb^*a^* + |a|^2bc\;) \end{array}}{(|a|^2 - |b|^2)(\;(|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2\;)} \right) \tag{171}$$

The above eqn (171) is the general unique solution that exists when, $(|a|^2 - |b|^2)((|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2) \neq 0$. The reader is invited to verify for himself, by substituting this formula for x back into the original equation, $ax - x^*b = c$, that this is indeed "a" solution to the given problem. On page 359 of [YT1] Tian gives the solution to this problem in his eqn (3.35), i.e. when $|a| \neq |b|$, in matrix form, using the components of the quaternions, and then declares that it is an unsolved problem how to write this solution in a formula composed by a, b, and c.

We propose that (171) is the general formula in the quaternion symbols a, b, and c, and not just "a" solution to the problem. We can show that the "special cases" given by Tian can be derived from this one general unique solution.

SPECIAL CASE 1: Condition, $|a| \neq |b|$ and $abc^* = c^*ab$. $\therefore \quad x = (|a|^2 - |b|^2)^{-1}(a^*c + bc^*)$

SPECIAL CASE 2: Condition, $a, b, \in \mathbb{H}$, $a \neq 0, b = ka$, $1 \neq k \in \mathbb{R}$. $\therefore \quad x = \dfrac{1}{(1-k^2)|a|^2}(a^*c + kac^*)$

SPECIAL CASE 3: Condition, $a, b \in \mathbb{H}$, $|a|^2 \neq |b|^2$, with, $c \in \mathbb{R}$. $\therefore \quad x = c(|a|^2 - |b|^2)^{-1}(a^* + b)$

SPECIAL CASE 4: Condition, $a, b \in \mathbb{H}$, with, $|a|^2 \neq |b|^2$, and, $c = ab$. $\therefore \quad x = c(|a|^2 - |b|^2)^{-1}(|b|^2a^* + |a|^2b)$

In the first of Tian's special cases, the condition is, $|a| \neq |b|$ and $abc^* = c^*ab$. Conjugating this latter condition, we have, $cb^*a^* = b^*a^*c$, also. Since, therefore, c^* commutes with ab, and c commutes with b^*a^*, we can move c^* and c to the right of all terms in the numerator of (171) and we obtain;

$$\therefore \qquad x = \left(\frac{\begin{matrix} (|a|^2 + |b|^2)(\ |a|^2a^*c + |a|^2bc^* + |b|^2a^*c + |b|^2bc^*\ ) \\ -((ab) + (ab)^*)(\ bb^*a^*c^* + babc^* + a^*b^*a^*c + a^*abc\ ) \end{matrix}}{(|a|^2 - |b|^2)(\ (|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2\ )} \right) \tag{172}$$

$$= \left(\frac{\begin{matrix} (|a|^2 + |b|^2)^2(a^*c + bc^*) \\ -((ab) + (ab)^*)(\ b(b^*a^* + ab)c^* + a^*(b^*a^* + ab)c\ ) \end{matrix}}{(|a|^2 - |b|^2)(\ (|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2\ )} \right) \tag{173}$$

$$= \left(\frac{((|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2)(a^*c + bc^*)}{(|a|^2 - |b|^2)(\ (|a|^2 + |b|^2)^2 - ((ab) + (ab)^*)^2\ )} \right) = \frac{a^*c + bc^*}{|a|^2 - |b|^2} \qquad \text{Q.E.D.} \tag{174}$$

In the second special case, Tian assumes the conditions from the first special case, although this is not stated clearly in his paper, so that, the previous, $abc^* = c^*ab$, being assumed, implies, $aac^* = c^*aa$, here also. Then, the simple substitution of $b = ka$, in eqn (174), leads directly to that unique solution;

$$x = \frac{a^*c + bc^*}{|a|^2 - |b|^2} = \frac{a^*c + kac^*}{|a|^2 - |ka|^2} = \frac{a^*c + kac^*}{(1-k^2)|a|^2} \qquad \text{Q.E.D} \tag{175}$$

In the third special case, the inhomogenous parameter, c, is a real valued variable, so that, again, the commutation relation, $abc^* = c^*ab$, holds, and we obtain (172) – (174), only this time, $c^* = c$, so we may reduce once again;

$$x = \frac{a^*c + bc^*}{|a|^2 - |b|^2} = \frac{a^*c + bc}{|a|^2 - |b|^2} = \frac{c(a^* + b)}{|a|^2 - |b|^2} \qquad \text{Q.E.D} \tag{176}$$

In Tian's final special case, since, $c = ab$, we have, $ab(ab)^* = abb^*a^* = a|b|^2a^*\ |a|^2|b|^2 = (ab)^*ab$, so again, the commutation condition, $abc^* = c^*ab$, is being assumed, and Tian's last three special cases are just sub-cases of his first special case, so that substituting $c = ab$ in (174) yields the last unique solution;

$$x = \frac{a^*c + bc^*}{|a|^2 - |b|^2} = \frac{a^*ab + b(ab)^*}{|a|^2 - |b|^2} = \frac{c(|a|^2b + |b|^2a^*)}{|a|^2 - |b|^2} \qquad \text{Q.E.D} \tag{177}$$

On page 354 of [YT1], in discussing his equation (3.6), $[\phi(a) - \tau(b)L]\hat{x} = 0$, Tian gives the formula for the determinant, $\det[\phi(a) - \tau(b)L] = (|a|^2 - |b|^2)|a^* + b|^2$. If we expand the factor, $|a^* + b|^2 = (a^* + b)^*(a^* + b) = (|a|^2 + |b|^2 + (ab) + (ab)^*)$, and give the denominator in eqn (171) the label λ, then we can compare these divisors;

$$\det[\phi(a) - \tau(b)L] = (|a|^2 - |b|^2)(|a|^2 + |b|^2 + (ab) + (ab)^*) \tag{178}$$

$$\lambda = (|a|^2 - |b|^2)(|a|^2 + |b|^2 + (ab) + (ab)^*)(|a|^2 + |b|^2 - (ab) - (ab)^*) \tag{179}$$

We have an extra factor, $((|a|^2+|b|^2)-((ab)+(ab)^*))$, in the divisor, which allows the numerator to be expressed in higher order terms; instead of p^4, the formula (171) is expressed in p^6 terms, but these 'parameters' can be reduced to p^4, by breaking the inhomogenous quaternion parameter c into its scalar and vector parts, $c = S(c) + V(c)$. This allow us to get rid of the extra factor in the divisor, and write,

$$x = \left(\frac{\begin{array}{l} +((|a|^2+|b|^2)+((ab)+(ab)^*))(a^*+b)S(c) \\ +(|a|^2a^* - |b|^2b - ((ab)+(ab)^*)(b-a^*))V(c) - a^*V(c)ab + bV(c)b^*a^* \end{array}}{(|a|^2-|b|^2)|a^*+b|^2} \right) \tag{180}$$

Of course, once we have broken up the c quaternion to enable us to continue the reduction of the expression form, and have found the ultimate solution, we can always "be clever" and put back in the whole quaternion c, by using the simple identities, $S(c) = (c+c^*)/2$, and, $V(c) = (c-c^*)/2$, if we really insist on seeing the pure form with whole quaternions, and so re-write the (171) solution;

$$x = \left(\frac{\begin{array}{l} +((|a|^2+|b|^2)+((ab)+(ab)^*))(a^*+b)\cdot\frac{1}{2}\cdot(c+c^*) \\ +(|a|^2a^* - |b|^2b - ((ab)+(ab)^*)(b-a^*))\cdot\frac{1}{2}\cdot(c-c^*) - a^*\cdot\frac{1}{2}\cdot(c-c^*)\cdot ab + b\cdot\frac{1}{2}\cdot(c-c^*)\cdot b^*a^* \end{array}}{(|a|^2-|b|^2)|a^*+b|^2} \right) \tag{181}$$

or some variation thereof.

III. CONCLUSIONS.

The above results indicate that *hand transformation*, $A \to A'$, may be as important as *conjugation*, $A \to A^*$, in empowering the calculus of quaternions to solve problems, and is probably a necessary tool required for working with quaternions. Conjugation is often thought of as a kind of hand transformation, given that, when the units, i, j, k, obey the right hand rule, $ij = +k$, their conjugates, i^*, j^*, k^*, obey the left hand rule, $i^*j^* = -k^*$, just like our left hand units, $i'j' = -k'$. However, as pointed out in our previous paper[1] [PJ2], there is a special distinguished left handed basis[11] corresponding to a given right handed basis, separate and distinct from the conjugate basis, that plays the role of true left hand. The difference is most clearly understood when considering the transformations induced by these numbers when they play the role of operators. Where a unit quaternion, $i = i_R$, produces a rotation, the conjugate quaternion, $i^* = i_R^*$, reverses that operation, but while the left hand quaternion, $i' = i_L$, also reverses the very same rotation, it goes further and induces a reflection in the plane perpendicular to the axis of rotation. In physical jargon, the conjugate * is linked to PARITY, which is a complete three axis space inversion, while the hand transform $'$ is linked to CHIRALITY, which is a plane mirror single axis inversion. The net result of $i^*i = i_R^*i_R$, is just the identity transformation, while the net result of $i'i = i_Li_R$, is a reflection in the jk-plane[12]. Therefore, although the idea of the conjugate includes considerations of left-handedness within the calculus of quaternions constructed from a right hand basis alone, it is really incomplete, and a distinct left hand basis needs to be included to complete both the geometric picture and for manipulation of expressions to solve purely algebraic problems. Now, the conjugate commutes with the original parameter, $Q^*Q = QQ^*$, and the hand transform also commutes, $Q'Q = QQ'$. But, while the conjugate only generally commutes with the *original quaternion* it transformed from, and closely related numbers, i.e. typically $Q^*P \neq PQ^*$, the hand transform will commute with all quaternions in the *original algebra*, $Q'P = PQ'$, $\forall\ P \in \mathbb{H}_R$. In this way, if we can think of the conjugate as a kind of hand transform, then the hand transform is a kind of conjugate, at least in respect to permutation of factors, except the true conjugate is restricted to local commutes, $Q^*\mathbb{S} = \mathbb{S}Q^*$, $\mathbb{S} \equiv \{P \mid P = \lambda Q + \beta;\ \lambda, \beta \in \mathbb{R}\} \subset \mathbb{H}_R$, while the hand transform is like a global conjugate, $Q'\mathbb{H}_R = \mathbb{H}_RQ'$, that commutes with all parameters in the original algebra. Thinking of these two marks, * and $'$, as *local* and *global* operators, respectively, helps to fix ideas on why we need two separate and distinct types of hand transformation operators—that both seem to produce left handed numbers—to complete the calculus of quaternions: the local conjugate helps to invert individual quaternions, and the global conjugate facilitates inversion of linear expressions of those quaternions, enabling us to solve linear equations.

SINGULAR SOLUTIONS. For the linear problems described in this paper, we have given only the "general solutions." These are the solutions obtained when the scalar denominator in the formulas for h^{-1} is non-zero, so that this inverse exists. As briefly discussed in[1] ([PJ2], pg. 27), there may be solutions that exist for the singular problems where h^{-1} itself does not exist, but in this paper we have not given any further treatment of the singular solutions.

$$T(q) = A_1 q B_1 + A_2 q B_2 + \cdots + A_n q B_n \tag{182}$$

Now every linear transformation, $T : (v_0, v_1, v_2, v_3) \mapsto (w_0, w_1, w_2, w_3)$, in real valued 4-dimensional space, can be represented by a 4×4 real matrix, $T \in M(\mathbb{R}, 4)$, and every such matrix can be written as an hexpe number, $h \in \mathbb{X}_n$. But, since every hexpe number, h, can also be written as the sum of pair products, $h = A_1 B_2' + \cdots + A_n B_n'$, with one right and one left handed quaternion, $A_k \in \mathbb{H}_R$, $B_k' \in \mathbb{H}_L$, in each pair, this means that the linear transformation can be written–(182)–as the sum of quaternion terms, $A_k q B_k$, using the right hand quaternion algebra, i.e. $A_k, B_k, q \in \mathbb{H}_R$. The four real variable parameters, v_k, $k = 0, 1, 2, 3$, are bundled up and tossed into a quaternion, $q_k = v_k$; and the general linear transformation is treated as a sequence of operations on quaternions, using both multiplication from the left and from the right simultaneously, within Hamilton's right handed algebra. The operator acting TO THE LEFT, $qB \equiv q \leftarrow B$, introduces left-handed actions in the transformation operation—while still working within this set of right handed parameters—and these left-actions are then converted into operator TO THE RIGHT actions, $B'\hat{q} \equiv B' \rightarrow \hat{q}$, using left-handed quaternions instead. This enables us to deal with both left and right acting operators within the same expressions, rather than the usual convention that tends to focus on just a one sided algebra. A general linear transformation is then seen as a set of *simultaneous* " left and right " actions on a central object, which is the subject of the transformation. This mimics known processes in physical phenomena—a rotation is produced by a simultaneous action of two equal but *opposite forces*—"acting on opposite sides"—of a central object, which is the subject of the torque. Many processes in the physical world can be interpreted by a transformation produced with two opposing forces—even a stationary object, by Netwton's first law, remains put, because action and reaction are equal and opposite. Now, when the inverse, h^{-1}, exists, this corresponds to the existence of the inverse matrix, T^{-1}, in $M(\mathbb{R}, 4)$, and these are the reversible linear transformations, $GL(\mathbb{R}, 4)$, known as affine transformations. The "general solutions" described in this paper, therefore, are useful for the study of the four dimensional affine geometry, and Singular solutions are only required for non-affine transformations.

HOMOGENEOUS SOLUTIONS. Special arbitrary solutions may also exist when $C = 0$, in (4), i.e. for the corresponding homogeneous problem, if the problem is singular, and these need to be added to the singular solution for $C \neq 0$. One example is given in[1] ([PJ2], pg. 27) as part of the singular solution discussed there. If the homogenous solution is q_H, and the singular solution is q_S, then the "complete solution" to the problem, in this case, is then: $q = q_S + q_H$.

COMMENT. In this paper, we have generally avoided making reference to the components of a quaternion, to it's particular basis elements, or even to the scalar and vector parts, in the process of manipulation of expressions[13]. We only illustrate the *scalar + vector* notational form in some results for comparison [$Pg.6$]. Hamilton thought [1] it was a deficiency [14] in the method whenever one had to break the quaternion down into its components $1, i, j, k$, or make reference to these components in the working out of solutions. (We still have to do this to treat the singular solutions, for example.) But, he felt comfortable splitting the quaternion up into its scalar and vector parts, $q = Sq + Vq$, an idea that was obviously superior to him because it made no specific reference to spatial coordinates (Which parallels our idea of space as homogeneous—no unique origin of coordinates—and isotropic—looking the same in all directions.) However, this very decomposition caused many, like Heaviside and Gibbs, to feel that quaternions were artificial and composed of the unnatural union of two separate algebras: an algebra of scalars and an algebra of vectors. They could not concieve of the quaternion as a necessarily complete object in its own right. So, they eventually deconstructed Hamilton's calculus to separate the parts they felt should be kept distinct for vector algebra. This has led the spacetime algebra to evolve down a particular path, culminating in modern Clifford algebra, where one aspect of geometric transformations is overly emphasized–*the rotation*–while another important aspect–*the scalings*–are wholey depreciated, in the mathematical description of nature's art. The two-hand quaternion algebra restors the balance between the rotation and the scaling transformations, in algebraic geometry, since it represents the complete linear transformation that describes all of affine geometry, without neglecting, nor overly emphasizing, any particular aspect!

We now take the position that it is sometimes also a deficiency in the method whenever one has to break the quaternion up into its scalar and vector parts, and write $q = Sq + Vq$, to effect a solution. And consider it superior when we can solve problems using q and q' instead, where $'$ is hand transformation, along with the usual q and q^* , where * is conjugation, since this treats the quaternion as a wholey complete entity in its own right. The right and left conjugate operators, $(\,\cdot\,)^{*R}$ and $(\,\cdot\,)^{*L}$, are seen only as *shorthand notation*, for what is really the application of the normal conjugate, $(\,\cdot\,)^*$, within bilinear expressions, useful in the construction of alternate multiplicative factors—i.e. instead of writing out those factors with explicit expressions using the normal * conjugate on a term by term basis—and are not to be thought of as new fundamental operators adding something more than the normal conjugate itself to the understanding of the theory of quaternion calculus.

THREE TERM AND QUARKS. Three term linear equations (3) could prove useful in modeling certain physical phenomena, like elementary particles called quarks, where the underlying structure inherent in the physical process is thought to consist of a triplet. There is something special about this linear problem in quaternions, since it seems to be the highest number of terms solvable with the algebraic method we started out with, before we have to split the conjugate $(\cdot)^*$ into right $(\cdot)^{*R}$ and left $(\cdot)^{*L}$ parts, and start combining different conjugated cubes, in order to effect solutions. The solutions become intricately more complicated for four terms and higher, and our "simpler method" introduced to solve these higher problems—which makes them tractable—is actually more complicated for three term and less, and so the label "simpler method" appears to be a misnomer when applied there. But, in fact, our initial method required guessing the factors to be applied to reduce the L.H.S to scalar values, and that is, indeed, more complicated, especially when we have to start combining different guesses to find the right construction. It is then simpler to use the direct method made available through the quaternion expansion of the adjoint matrix to walk a known and sure path to the solution, for even though more algebraic steps are involved, there is no guesswork. The four term linear equation is also somewhat special, in that it is the last to introduce new types of expression blocks in the reduced formulas for the inverse, h^{-1} , and the solution, q. We can immediately write down the reduced solutions to the n-term problem, by simply inspecting the 4-term formula and extending the form of the expression blocks found there to n-indicies by following the pattern of the terms. Beyond the "three" and "four" term, however, the further n-term problems seem to add nothing essentially new, apart from extending the index count in the solution.

TENSOR PRODUCT ALGEBRAS. Our algebra introduces the left hand quaternions, the concept of hand transformation, the idea of the partial conjugates, $(\cdot)^{*R}$ and $(\cdot)^{*L}$, and various conjugated squares and conjugated cubes, to evolve a method for solving linear problems. However, because of isomorphism, it is quite feasible to erect a double right hand algebra instead, with partial conjugates $(\cdot)^{*R_1}$ and $(\cdot)^{*R_2}$, and develop a parallel method for solving these linear problems by enhancing the algebra of the tensor product $\mathbb{H}\otimes\mathbb{H}$. In such a case, the concept of *hand transformation* would appear to be non-essential, being replaced by some other way of recognising the conversion between differing right hands instead. But, the double right hand algebra seems non-intuitive to us, since it currently appears to lack the power to contribute to physical insights, and we have therefore chosen the more balanced two-hand $\mathbb{H}'\otimes\mathbb{H}$ instead, because of this very appeal in anticipated applications to physical problems. However, adapting these new methods to the double right hand $\mathbb{H}\otimes\mathbb{H}$, could then enable the development of similar constructions and methods of approach in solving problems for even higher order tensor product algebras, such as third order, $\mathbb{H}^3\equiv\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$, extending to $\mathbb{H}^n\equiv\mathbb{H}\otimes\mathbb{H}\otimes\cdots\otimes\mathbb{H}$. So, this obvious alternative may also merit some consideration.

QUATERNION MATRIX EXPANSIONS. As shown in the APPENDIX , we can construct a "quaternion expansion" for the adjoint and determinant of any 4×4 real matrix, and thus effectively carry out matrix operations using quaternion computations. This is not a particularly efficient way of doing matrix algebra, especially when the real matrix coefficients, a_{uv}, are given. But, it does allow us to work out matrix algebra entirely in quaternions, without referencing those matrix components, when an alternate quaternion representation is given—either the bilinear form, or hexpe basis form. The quaternion expansions given herein are independent of the particular 16-d hexpe basis chosen to represent a matrix, and so are more general formulas than those that make specific reference to our particular hexpentaquaternion basis matricies established in our previous paper [PJ2]. The equivalence of the two-hand quaternion to matrix algebra is more important in the establishment of methods of solutions to purely quaternion problems, where we seek to reckon with the non-abelian parameters without breaking the quaternion out into its components. Here we use the fact that the two-hand quaternion algebra is based on solid foundations, with a large body of established work behind it, in the corresponding matrix algebra, so that we can be confident in our new results without alot of additional theorem proving being first required. We use this, for example, to establish that the solution we have found includes all possible general unique solutions, and to allocate the variety of special case solutions—that generally otherwize populate quaternion problems—to the special category of the singular case. The new approach clearly leads us easily to the solution of previously unsolved quaternion problems, specifically the problem mentioned by Tian of how to compose a formula in quaternions for the unique solutions that are already known to exist, which problems remained unsolved simply because of the shear difficulty of the task to establish certain solutions with components first, and *then* seek to gather those components back into the quaternions from which they came, in order to view the results in whole quaternion parameter format.

A.

APPENDIX

CONJUGATED CUBES.

In basis component format, the inverse of a general hexpe number, $h \in \mathbb{X}_n$, is given by, $h^{-1} \in \mathbb{X}_n$, where,

$$\begin{aligned} h = h_0 \cdot \boldsymbol{E} &+ h_{M1} \cdot \boldsymbol{I}_M + h_{M2} \cdot \boldsymbol{J}_M + h_{M3} \cdot \boldsymbol{K}_M \\ + h_{R1} \cdot \boldsymbol{I}_R &+ h_{L1} \cdot \boldsymbol{I}_L + h_{A1} \cdot \boldsymbol{I}_A + h_{Z1} \cdot \boldsymbol{I}_Z \\ + h_{R2} \cdot \boldsymbol{J}_R &+ h_{L2} \cdot \boldsymbol{J}_L + h_{A2} \cdot \boldsymbol{J}_A + h_{Z2} \cdot \boldsymbol{I}_Z \\ + h_{R3} \cdot \boldsymbol{K}_R &+ h_{L3} \cdot \boldsymbol{K}_L + h_{A3} \cdot \boldsymbol{K}_A + h_{Z3} \cdot \boldsymbol{K}_Z \end{aligned} \qquad \begin{aligned} h^{-1} = (w_0 \cdot \boldsymbol{E} &+ w_{M1} \cdot \boldsymbol{I}_M + w_{M2} \cdot \boldsymbol{J}_M + w_{M3} \cdot \boldsymbol{K}_M \\ + w_{R1} \cdot \boldsymbol{I}_R &+ w_{L1} \cdot \boldsymbol{I}_L + w_{A1} \cdot \boldsymbol{I}_A + w_{Z1} \cdot \boldsymbol{I}_Z \\ + w_{R2} \cdot \boldsymbol{J}_R &+ w_{L2} \cdot \boldsymbol{J}_L + w_{A2} \cdot \boldsymbol{J}_A + w_{Z2} \cdot \boldsymbol{I}_Z \\ + w_{R3} \cdot \boldsymbol{K}_R &+ w_{L3} \cdot \boldsymbol{K}_L + w_{A3} \cdot \boldsymbol{K}_A + w_{Z3} \cdot \boldsymbol{K}_Z)/d \end{aligned} \qquad \text{(A-1)}$$

$$\begin{aligned} d = {} & h_0 w_0 + h_{M1} w_{M1} + h_{M2} w_{M2} + h_{M3} w_{M3} + h_{A1} w_{A1} + h_{A2} w_{A2} + h_{A3} w_{A3} + h_{Z1} w_{Z1} + h_{Z2} w_{Z2} + h_{Z3} w_{Z3} \\ & - h_{R1} w_{R1} - h_{R2} w_{R2} - h_{R3} w_{R3} - h_{L1} w_{L1} - h_{L2} w_{L2} - h_{L3} w_{L3} \end{aligned} \qquad \text{(A-2)}$$

The formulas for the weights, $w_k \in \mathbb{R}$, are given previously in [1] (TABLE T. 3-IV) of [PJ2], and are cubic in the coefficients of h, i.e. $w_k = O(h^3)$. So, when h is written in two-hand bilinear form, i.e $h = A_1 B_1' + A_2 B_2' + \cdots + A_n B_n'$, $A_k \in \mathbb{H}_R, B_k' \in \mathbb{H}_L$, the weights are then simultaneously cubic in the right and left handed quaternion factors, $w_k = O(A^3 B^3)$. This is always the case, regardless of how many terms n comprise the bilinear expression. We never need to go higher than the cubic to construct the corresponding inverse formula for the bilinear hexpe number. The denominator is a scalar of quartic order in the coefficients of h, i.e. $d = O(h^4)$, and thus this scalar is bi-quartic in the corresponding right and left hand quaternions, $d = O(A^4 B^4)$. Hence, knowing this, we then seek to write the inverse in the bi-cubic form, with numerator terms $A_.^* A_. A_.^* B_.'^* B_.' B_.'^*$ etc.., and some scalar denominator $\lambda \in \mathbb{R}$, and so write,

$$h^{-1} = \frac{\sum A_.^* A_. A_.^* B_.'^* B_.' B_.'^*}{\lambda}, \qquad A_. \in \mathbb{H}_R,\ B_.' \in \mathbb{H}_L,\ \lambda \in \mathbb{R}. \qquad \text{(A-3)}$$

instead of expressing the inverse in the basis components. To obtain this formula, we must construct it from variations of the conjugated cubes, e.g. $h^{*S} h^{*T} h^{*U}$, where, the $(\cdot)^{*S}$, $(\cdot)^{*T}$, and $(\cdot)^{*U}$, are conjugations taken from the three possible conjugates, $(\cdot)^{*R}$, $(\cdot)^{*L}$, and $(\cdot)^{*}$, i.e. the right conjugate, left conjugate, and regular congugate, or the unconjugated h taken as a candidate factor. In basis component terms, these conjugates are defined by;

$$\begin{aligned} h &= h_0 + h_{R1}\boldsymbol{i} + h_{R2}\boldsymbol{j} + h_{R3}\boldsymbol{k} + h_{L1}\boldsymbol{i}' + h_{L2}\boldsymbol{j}' + h_{L3}\boldsymbol{k}' && \text{(A-4)} \\ &+ h_{M1}\boldsymbol{ii}' + h_{M2}\boldsymbol{jj}' + h_{M3}\boldsymbol{kk}' + h_{A1}\boldsymbol{jk}' + h_{A2}\boldsymbol{ki}' + h_{A3}\boldsymbol{ij}' + h_{Z1}\boldsymbol{kj}' + h_{Z2}\boldsymbol{ik}' + h_{Z3}\boldsymbol{ji}' \\ h^{*R} &= h_0 - h_{R1}\boldsymbol{i} - h_{R2}\boldsymbol{j} - h_{R3}\boldsymbol{k} + h_{L1}\boldsymbol{i}' + h_{L2}\boldsymbol{j}' + h_{L3}\boldsymbol{k}' && \text{(A-5)} \\ &- h_{M1}\boldsymbol{ii}' - h_{M2}\boldsymbol{jj}' - h_{M3}\boldsymbol{kk}' - h_{A1}\boldsymbol{jk}' - h_{A2}\boldsymbol{ki}' - h_{A3}\boldsymbol{ij}' - h_{Z1}\boldsymbol{kj}' - h_{Z2}\boldsymbol{ik}' - h_{Z3}\boldsymbol{ji}' \\ h^{*L} &= h_0 + h_{R1}\boldsymbol{i} + h_{R2}\boldsymbol{j} + h_{R3}\boldsymbol{k} - h_{L1}\boldsymbol{i}' - h_{L2}\boldsymbol{j}' - h_{L3}\boldsymbol{k}' && \text{(A-6)} \\ &- h_{M1}\boldsymbol{ii}' - h_{M2}\boldsymbol{jj}' - h_{M3}\boldsymbol{kk}' - h_{A1}\boldsymbol{jk}' - h_{A2}\boldsymbol{ki}' - h_{A3}\boldsymbol{ij}' - h_{Z1}\boldsymbol{kj}' - h_{Z2}\boldsymbol{ik}' - h_{Z3}\boldsymbol{ji}' \\ h^{*} &= h_0 - h_{R1}\boldsymbol{i} - h_{R2}\boldsymbol{j} - h_{R3}\boldsymbol{k} - h_{L1}\boldsymbol{i}' - h_{L2}\boldsymbol{j}' - h_{L3}\boldsymbol{k}' && \text{(A-7)} \\ &+ h_{M1}\boldsymbol{ii}' + h_{M2}\boldsymbol{jj}' + h_{M3}\boldsymbol{kk}' + h_{A1}\boldsymbol{jk}' + h_{A2}\boldsymbol{ki}' + h_{A3}\boldsymbol{ij}' + h_{Z1}\boldsymbol{kj}' + h_{Z2}\boldsymbol{ik}' + h_{Z3}\boldsymbol{ji}' \end{aligned}$$

where we have used the alternate representation of the basis elements, derived from the right hand quaternion basis $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, by using hand transformation $'$ and right left binary products, $\boldsymbol{I}_M = \boldsymbol{ii}'$, etc.. since this makes it easier to see, at a glance, why the right conjugate also conjugates the M-A-Z numbers, but the regular conjugate does not flip those signs. The regular conjugate $(\cdot)^*$ acts on both right handed and left handed components simultaneously, and is therefore equivalent to,

$$h^* = (h^{*R})^{*L} = (h^{*L})^{*R} \qquad \text{(A-8)}$$

so, when the one hand conjugate flips the sign on M-A-Z, the other hand conjugate flips it back, resulting in unchanged sign for these basis components. When written in bilinear form, the conjugates of h are;

$$\begin{aligned} h &= A_1 B_1' + A_2 B_2' + \cdots + A_n B_n' && \text{(A-9)} \\ h^{*R} &= A_1^* B_1' + A_2^* B_2' + \cdots + A_n^* B_n' && \text{(A-10)} \\ h^{*L} &= A_1 B_1'^* + A_2 B_2'^* + \cdots + A_n B_n'^* && \text{(A-11)} \\ h^{*} &= A_1^* B_1'^* + A_2^* B_2'^* + \cdots + A_n^* B_n'^* && \text{(A-12)} \end{aligned}$$

In a previous paper [PJ3] we also introduced the operators, $R(\cdot)$ and $L(\cdot)$, that extract the *right pure quaternion* and *left pure quaternion* of an hexpe number, respectively. This intentionally imitates Hamilton's use of the operator $V(\cdot)$ to extract the vector part–i.e. the pure quaternion–of a one hand quaternion. With our notation, we observe,

$$h + h^{*L} = 2S(h) + 2R(h) \tag{A-13}$$

$$h + h^{*R} = 2S(h) + 2L(h) \tag{A-14}$$

$$h + h^{*} = 2S(h) + 2M(h) + 2A(h) + 2Z(h) = 2h - 2R(h) - 2L(h) \tag{A-15}$$

where, $S(\cdot)$, is the usual scalar part operator from Hamilton, and we now introduce, $M(\cdot)$, $A(\cdot)$, and $Z(\cdot)$, for the pure META, ALPHA, and ZETA, parts, i.e. the M-A-Z, of the two-hand quaternion.

$$(h + h^{*L})^* = h^* + (h^{*L})^* = h^* + h^{*R} = (h^{*L} + h)^{*R} = 2S(h) - 2R(h) \tag{A-16}$$

$$(h + h^{*R})^* = h^* + (h^{*R})^* = h^* + h^{*L} = (h^{*R} + h)^{*L} = 2S(h) - 2L(h) \tag{A-17}$$

$$(h + h^{*L}) + (h + h^{*L})^* = h + h^* + h^{*L} + h^{*R} = 4S(h) \tag{A-18}$$

That is, the average of the four conjugated states of h is its scalar component $S(h)$, and we can use this to extract the scalar part of any hexpe number. Similarily, we can extract other parts of the two-hand quaternion,

$$S(h) = (h + h^* + h^{*R} + h^{*L})/4 \tag{A-19}$$

$$R(h) = (h - h^* - h^{*R} + h^{*L})/4 \tag{A-20}$$

$$L(h) = (h - h^* + h^{*R} - h^{*L})/4 \tag{A-21}$$

$$M(h) + A(h) + Z(h) = (h + h^* - h^{*R} - h^{*L})/4 \tag{A-22}$$

Conjugated Squares. The following table gives the various possible conjugations of the square forms, $\sim h^2$, we find,

hh	$(h^{*L}h)^{*R}$	$h^{*R}h^{*R}$	$.(h^{*R}h^*)^{*R}$
	$(hh^*)^{*R}$	$h^{*L}h^{*L}$	$.(h^*h^{*R})^{*R}$
$(hh)^{*R}$	$(h^*h)^{*R}$		$.(h^{*L}h^*)^{*R}$
hh^{*R}	$(hh^{*R})^{*L}$	$h^{*R}h^{*L}$	$.(h^*h^{*L})^{*R}$
$h^{*R}h$	$(h^{*R}h)^{*L}$	$h^{*L}h^{*R}$	
$(hh)^{*L}$	$(hh^{*L})^{*L}$	$h^{*R}h^*$	
hh^{*L}	$(h^{*L}h)^{*L}$	h^*h^{*R}	$(h^{*R}h^{*R})^{*L}$
$h^{*L}h$	$.(hh^*)^{*L}$	$h^{*L}h^*$	$(h^{*L}h^{*L})^{*L}$
$(hh)^*$	$.(h^*h)^{*L}$	h^*h^{*L}	$.(h^*h^*)^{*L}$
hh^*			$(h^{*R}h^{*L})^{*L}$
h^*h		$.(h^{*R}h^{*R})^{*R}$	$(h^{*L}h^{*R})^{*L}$
		$.(h^{*L}h^{*L})^{*R}$	$.(h^{*R}h^*)^{*L}$
$(hh^{*R})^{*R}$		$.(h^*h^*)^{*R}$	$.(h^*h^{*R})^{*L}$
$(h^{*R}h)^{*R}$		$.(h^{*R}h^{*L})^{*R}$	$.(h^{*L}h^*)^{*L}$
$(hh^{*L})^{*R}$		$.(h^{*L}h^{*R})^{*R}$	$.(h^*h^{*L})^{*L}$

These are the zero, single, double, and triple conjugated squares. All other conjugated squares reduce to one of these. Some of these are not unique, and are equivalent to another in the same table, but the equivalence is non-obvious. For example, $(h^{*L}h^{*R})^*$, is a triple conjugated square not included in the table, because it's obvious that, $(h^{*L}h^{*R})^* = (h^{*R})^*(h^{*L})^* = h^{*L}h^{*R}$, which is a double conjugated square that is already included. The regular conjugate obeys the rule, $(gh)^* = h^*g^*$, which is presumed familiar to the reader; however, partial conjugates do not, $(gh)^{*R} \neq h^{*R}g^{*R}$, and, $(gh)^{*L} \neq h^{*L}g^{*L}$, in general, for, $g, h \in \mathbb{X}_n$. Less obvious, is that, $(h^*h^*)^{*R} = ((hh)^*)^{*R} = (hh)^{*L}$, so both are included; one being marked by a preceeding dot "." to indicate that it is a duplicate of another table element. Similarly, $(h^*h)^{*L} = ((h^*h)^*)^{*L} = (h^*h)^{*R}$, and, $(h^{*L}h)^{*L} = ((h^{*L}h)^*)^{*R} = (h^*h^{*R})^{*R}$, etc... There are 32 unique conjugated squares in this table.

Conjugated Cubes. The following are a few useful conjugated cube forms, $\sim h^3$, which we express in terms of their basis components, using the weights, w_k , from the known inverse, h^{-1} , to illustrate where they agree with that inverse's numerator, and where they differ. The goal is to find those combinations of conjugated cubes that can construct the numerator for h^{-1}, since this will reveal to us the sequence of conjugations and arithmetic operations required to transform the L-H-S of a general linear problem into a scalar value, enabling us to invert these equations.

$$
\begin{aligned}
&(h^{*R}h)^{*L}h^{*R} = \\
&\quad +1\cdot(\ w_0 + 8(+h_{M1}h_{M2}h_{M3} + h_{A1}h_{A2}h_{A3} + h_{Z1}h_{Z2}h_{Z3} - h_{M1}h_{A1}h_{Z1} - h_{M2}h_{A2}h_{Z2} - h_{M3}h_{A3}h_{Z3})\ ) \\
&\quad +I_R\cdot(\ w_{R1} + 8(+h_{L1}h_{M2}h_{M3} + h_{L2}h_{A1}h_{A2} + h_{L3}h_{Z1}h_{Z3} - h_{L1}h_{A1}h_{Z1} - h_{L2}h_{M3}h_{Z3} - h_{L3}h_{M2}h_{A2})\ ) \\
&\quad +J_R\cdot(\ w_{R2} + 8(+h_{L1}h_{Z1}h_{Z2} + h_{L2}h_{M1}h_{M3} + h_{L3}h_{A2}h_{A3} - h_{L1}h_{M3}h_{A3} - h_{L2}h_{A2}h_{Z2} - h_{L3}h_{M1}h_{Z1})\ ) \\
&\quad +K_R\cdot(\ w_{R3} + 8(+h_{L1}h_{A1}h_{A3} + h_{L2}h_{Z2}h_{Z3} + h_{L3}h_{M1}h_{M2} - h_{L1}h_{M2}h_{Z2} - h_{L2}h_{M1}h_{A1} - h_{L3}h_{A3}h_{Z3})\ ) \\
&\quad +I_L\cdot w_{L1} \\
&\quad +J_L\cdot w_{L2} \\
&\quad +K_L\cdot w_{L3} \\
&\quad +I_M\cdot w_{M1} \\
&\quad +J_M\cdot w_{M2} \\
&\quad +K_M\cdot w_{M3} \\
&\quad +I_A\cdot w_{A1} \\
&\quad +J_A\cdot w_{A2} \\
&\quad +K_A\cdot w_{A3} \\
&\quad +I_Z\cdot w_{Z1} \\
&\quad +J_Z\cdot w_{Z2} \\
&\quad +K_Z\cdot w_{Z3}
\end{aligned}
\tag{C-1}
$$

$$
\begin{aligned}
&(h^{*L}h)^{*R}h^{*L} = \\
&\quad +1\cdot(\ w_0 + 8(+h_{M1}h_{M2}h_{M3} + h_{A1}h_{A2}h_{A3} + h_{Z1}h_{Z2}h_{Z3} - h_{M1}h_{A1}h_{Z1} - h_{M2}h_{A2}h_{Z2} - h_{M3}h_{A3}h_{Z3})\ ) \\
&\quad +I_R\cdot w_{R1} \\
&\quad +J_R\cdot w_{R2} \\
&\quad +K_R\cdot w_{R3} \\
&\quad +I_L\cdot(\ w_{L1} + 8(+h_{R1}h_{M2}h_{M3} + h_{R2}h_{Z1}h_{Z2} + h_{R3}h_{A1}h_{A3} - h_{R1}h_{A1}h_{Z1} - h_{R2}h_{M3}h_{A3} - h_{R3}h_{M2}h_{Z2})\ ) \\
&\quad +J_L\cdot(\ w_{L2} + 8(+h_{R1}h_{A1}h_{A2} + h_{R2}h_{M1}h_{M3} + h_{R3}h_{Z2}h_{Z3} - h_{R1}h_{M3}h_{Z3} - h_{R2}h_{A2}h_{Z2} - h_{R3}h_{M1}h_{A1})\ ) \\
&\quad +K_L\cdot(\ w_{L3} + 8(+h_{R1}h_{Z1}h_{Z3} + h_{R2}h_{A2}h_{A3} + h_{R3}h_{M1}h_{M2} - h_{R1}h_{M2}h_{A2} - h_{R2}h_{M1}h_{Z1} - h_{R3}h_{A3}h_{Z3})\ ) \\
&\quad +I_M\cdot w_{M1} \\
&\quad +J_M\cdot w_{M2} \\
&\quad +K_M\cdot w_{M3} \\
&\quad +I_A\cdot w_{A1} \\
&\quad +J_A\cdot w_{A2} \\
&\quad +K_A\cdot w_{A3} \\
&\quad +I_Z\cdot w_{Z1} \\
&\quad +J_Z\cdot w_{Z2} \\
&\quad +K_Z\cdot w_{Z3}
\end{aligned}
\tag{C-2}
$$

These are the particular conjugated cubes that each independently solve the "one term" and "two term" linear problems. Many components happen to be identical with the h^{-1} numerator, but not all. Some components have " extensions ", which are—in these two cubes—expressions that are of the form $+8(\ \cdots\ )$ shown. These extensions happen to evaluate to zero when the hexpe number is two term or less, i.e $h = A_1B_1'$, or, $h = (A_1B_1' + A_2B_2')$. But, for three term, and higher bilinear h forms, the extensions contribute a non-vanishing value, so we cannot simply use these conjugated cubes for the inverse's numerator as shown in (102). We can, however, extract the matching parts of these cubes and combine them. If we let, $h^{-1} = \eta(w)/\lambda$, where $\eta(w)$ is the required numerator, and, $\lambda \in \mathbb{R}$, then;

$$L((h^{*R}h)^{*L}h^{*R}) \ = \ +w_{L1}\boldsymbol{I}_L + w_{L2}\boldsymbol{J}_L + w_{L3}\boldsymbol{K}_L \tag{A-23}$$

$$R((h^{*L}h)^{*R}h^{*L}) \ = \ +w_{R1}\boldsymbol{I}_R + w_{R2}\boldsymbol{J}_R + w_{R3}\boldsymbol{K}_R \tag{A-24}$$

$$
\begin{aligned}
&(h^{*R}h)^{*L}h^{*R} - S((h^{*R}h)^{*L}h^{*R}) - R((h^{*R}h)^{*L}h^{*R}) \ = \\
&+w_{L1}\boldsymbol{I}_L + w_{L2}\boldsymbol{J}_L + w_{L3}\boldsymbol{K}_L + w_{M1}\boldsymbol{I}_M + w_{M2}\boldsymbol{J}_M + w_{M3}\boldsymbol{K}_M + w_{A1}\boldsymbol{I}_A + w_{A2}\boldsymbol{J}_A + w_{A3}\boldsymbol{K}_A + w_{Z1}\boldsymbol{I}_Z + w_{Z2}\boldsymbol{J}_Z + w_{Z3}\boldsymbol{K}_Z
\end{aligned}
\tag{A-25}
$$

$$
\begin{aligned}
&(h^{*L}h)^{*R}h^{*L} - S((h^{*L}h)^{*R}h^{*L}) - L((h^{*L}h)^{*R}h^{*L}) \ = \\
&+w_{R1}\boldsymbol{I}_R + w_{R2}\boldsymbol{J}_R + w_{R3}\boldsymbol{K}_R + w_{M1}\boldsymbol{I}_M + w_{M2}\boldsymbol{J}_M + w_{M3}\boldsymbol{K}_M + w_{A1}\boldsymbol{I}_A + w_{A2}\boldsymbol{J}_A + w_{A3}\boldsymbol{K}_A + w_{Z1}\boldsymbol{I}_Z + w_{Z2}\boldsymbol{J}_Z + w_{Z3}\boldsymbol{K}_Z
\end{aligned}
\tag{A-26}
$$

So, by combining just these two cubes, we obtain an exact match to the *vector part* of the numerator, $\eta(w) = \lambda h^{-1}$.

$$\lambda h^{-1} - S(\lambda h^{-1}) \;=\; (h^{*R}h)^{*L}h^{*R} - S((h^{*R}h)^{*L}h^{*R}) - R((h^{*R}h)^{*L}h^{*R}) + R((h^{*L}h)^{*R}h^{*L}) \tag{A-27}$$

$$\lambda h^{-1} - S(\lambda h^{-1}) \;=\; (h^{*L}h)^{*R}h^{*L} - S((h^{*L}h)^{*R}h^{*L}) - L((h^{*L}h)^{*R}h^{*L}) + L((h^{*R}h)^{*L}h^{*R}) \tag{A-28}$$

There are two ways to do this, starting with either cube, $(h^{*R}h)^{*L}h^{*R}$ or $(h^{*L}h)^{*R}h^{*L}$, extracting the components that differ because of extensions, and inserting the corresponding matching parts from the complementary cube, using the operators, $R(\,\cdot\,), L(\,\cdot\,)$, and $S(\,\cdot\,)$. The problem is that we can't extract the w_0, which is the scalar part of λh^{-1}, using just these two cubes, since the cubes have the same extension term on their scalar component. In fact, all *triple conjugated cubes*, constructed with the pair, $(\,\cdot\,)^{*R}$ and $(\,\cdot\,)^{*L}$, of partial conjugates, have identical scalar values, and contain the very same extension term again. We need to consider *double conjugated cubes*, like $h^{*R}hh^{*L}$, for example, to obtain a different scalar component extension term, so that we can construct an arithmetic expression that effectively isolates and extracts the w_0. This could also be done with a triple conjugated cube, where the regular conjugate $(\,\cdot\,)^*$ is one of the three conjugates, e.g. $(h^{*R}hh^{*L})^* = h^{*L}h^*h^{*R}$. So, this latter will also suffice.

$$\begin{aligned}
h^{*R}hh^{*L} = &\\
+1 \cdot (\, & w_0 - 4(+h_{M1}h_{M2}h_{M3} + h_{A1}h_{A2}h_{A3} + h_{Z1}h_{Z2}h_{Z3} - h_{M1}h_{A1}h_{Z1} - h_{M2}h_{A2}h_{Z2} - h_{M3}h_{A3}h_{Z3})\,)\\
+I_R \cdot (\, & -w_{R1} - 4(+h_{L1}h_{M2}h_{M3} + h_{L2}h_{A1}h_{A2} + h_{L3}h_{Z1}h_{Z3} - h_{L1}h_{A1}h_{Z1} - h_{L2}h_{M3}h_{Z3} - h_{L3}h_{M2}h_{A2})\,)\\
+J_R \cdot (\, & -w_{R2} - 4(+h_{L1}h_{Z1}h_{Z2} + h_{L2}h_{M1}h_{M3} + h_{L3}h_{A2}h_{A3} - h_{L1}h_{M3}h_{A3} - h_{L2}h_{A2}h_{Z2} - h_{L3}h_{M1}h_{Z1})\,)\\
+K_R \cdot (\, & -w_{R3} - 4(+h_{L1}h_{A1}h_{A3} + h_{L2}h_{Z2}h_{Z3} + h_{L3}h_{M1}h_{M2} - h_{L1}h_{M2}h_{Z2} - h_{L2}h_{M1}h_{A1} - h_{L3}h_{A3}h_{Z3})\,)\\
+I_L \cdot (\, & -w_{L1} - 4(+h_{R1}h_{M2}h_{M3} + h_{R2}h_{Z1}h_{Z2} + h_{R3}h_{A1}h_{A3} - h_{R1}h_{A1}h_{Z1} - h_{R2}h_{M3}h_{A3} - h_{R3}h_{M2}h_{Z2})\,)\\
+J_L \cdot (\, & -w_{L2} - 4(+h_{R1}h_{A1}h_{A2} + h_{R2}h_{M1}h_{M3} + h_{R3}h_{Z2}h_{Z3} - h_{R1}h_{M3}h_{Z3} - h_{R2}h_{A2}h_{Z2} - h_{R3}h_{M1}h_{A1})\,)\\
+K_L \cdot (\, & -w_{L3} - 4(+h_{R1}h_{Z1}h_{Z3} + h_{R2}h_{A2}h_{A3} + h_{R3}h_{M1}h_{M2} - h_{R1}h_{M2}h_{A2} - h_{R2}h_{M1}h_{Z1} - h_{R3}h_{A3}h_{Z3})\,)\\
+\,I_M \cdot & \, w_{M1} \cdots\\
+J_M \cdot & \, w_{M2} \cdots\\
+K_M \cdot & \, w_{M3} \cdots\\
+I_A \cdot & \, w_{A1} \cdots\\
+J_A \cdot & \, w_{A2} \cdots\\
+K_A \cdot & \, w_{A3} \cdots\\
+I_Z \cdot & \, w_{Z1} \cdots\\
+J_Z \cdot & \, w_{Z2} \cdots\\
+K_Z \cdot & \, w_{Z3} \cdots
\end{aligned} \tag{C-3}$$

Since we're only interested in the scalar component, the the vector parts of this conjugated cube don't matter. We show the first few extension terms only; M-A-Z extensions are indicated by $\cdots$ instead. By doubling this cube's scalar component and adding to the scalar part of either of the previous two cubes we obtain $3w_0$; then, divide by 3 to obtain w_0. We can now write the complete λh^{-1} numerator term in various ways, two of which are;

$$\begin{aligned}
\lambda h^{-1} \;=\;& (h^{*R}h)^{*L}h^{*R} - S((h^{*R}h)^{*L}h^{*R}) - R((h^{*R}h)^{*L}h^{*R}) + R((h^{*L}h)^{*R}h^{*L}) \qquad \text{(A-29)}\\
&+ \;(\; S((h^{*R}h)^{*L}h^{*R}) + 2S((h^{*R}hh^{*L})\;)/3\\
=\;& (h^{*R}h)^{*L}h^{*R} + R(\,(h^{*L}h)^{*R}h^{*L} - (h^{*R}h)^{*L}h^{*R}\,) + S(\,h^{*R}hh^{*L} - (h^{*R}h)^{*L}h^{*R}\,)\cdot(2/3) \qquad \text{(A-30)}
\end{aligned}$$

OR ,

$$\begin{aligned}
\lambda h^{-1} \;=\;& (h^{*L}h)^{*R}h^{*L} - S((h^{*L}h)^{*R}h^{*L}) - L((h^{*L}h)^{*R}h^{*L}) + L((h^{*R}h)^{*L}h^{*R}) \qquad \text{(A-31)}\\
&+ \;(\; S((h^{*L}h)^{*R}h^{*L}) + 2S((h^{*R}hh^{*L})\;)/3\\
=\;& (h^{*L}h)^{*R}h^{*L} + L(\,(h^{*R}h)^{*L}h^{*R} - (h^{*L}h)^{*R}h^{*L}\,) + S(\,h^{*R}hh^{*L} - (h^{*L}h)^{*R}h^{*L}\,)\cdot(2/3) \qquad \text{(A-32)}
\end{aligned}$$

Expanding the part operators using (A-19)-(A-22), we have,

$$\begin{aligned}
\lambda h^{-1} &= (h^{*R}h)^{*L}h^{*R} \\
&- (((h^{*R}h)^{*L}h^{*R}) + ((h^{*R}h)^{*L}h^{*R})^{*} + ((h^{*R}h)^{*L}h^{*R})^{*R} + ((h^{*R}h)^{*L}h^{*R})^{*L}) \cdot (2/12) \\
&- (((h^{*R}h)^{*L}h^{*R}) - ((h^{*R}h)^{*L}h^{*R})^{*} - ((h^{*R}h)^{*L}h^{*R})^{*R} + ((h^{*R}h)^{*L}h^{*R})^{*L}) \cdot (1/4) \\
&+ (((h^{*L}h)^{*R}h^{*L}) - ((h^{*L}h)^{*R}h^{*L})^{*} - ((h^{*L}h)^{*R}h^{*L})^{*R} + ((h^{*L}h)^{*R}h^{*L})^{*L}) \cdot (1/4) \\
&+ ((h^{*R}hh^{*L}) + (h^{*R}hh^{*L})^{*} + (h^{*R}hh^{*L})^{*R} + (h^{*R}hh^{*L})^{*L}) \cdot (2/12)
\end{aligned} \tag{A-33}$$

$$\begin{aligned}
\lambda h^{-1} &= (h^{*R}h)^{*L}h^{*R} \cdot \frac{7}{12} + ((h^{*R}h)^{*L}h^{*R})^{*} \cdot \frac{1}{12} + ((h^{*R}h)^{*L}h^{*R})^{*R} \cdot \frac{1}{12} - ((h^{*R}h)^{*L}h^{*R})^{*L} \cdot \frac{5}{12} \\
&+ (h^{*L}h)^{*R}h^{*L} \cdot \frac{3}{12} - ((h^{*L}h)^{*R}h^{*L})^{*} \cdot \frac{3}{12} - ((h^{*L}h)^{*R}h^{*L})^{*R} \cdot \frac{3}{12} + ((h^{*L}h)^{*R}h^{*L})^{*L} \cdot \frac{3}{12} \\
&+ h^{*R}hh^{*L} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*R} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*L} \cdot \frac{2}{12}
\end{aligned} \tag{A-34}$$

$$\therefore \quad h^{-1} = \left(\frac{\begin{array}{l} 7 \cdot (h^{*R}h)^{*L}h^{*R} + ((h^{*R}h)^{*L}h^{*R})^{*} + ((h^{*R}h)^{*L}h^{*R})^{*R} - 5 \cdot ((h^{*R}h)^{*L}h^{*R})^{*L} \\ + 3 \cdot (h^{*L}h)^{*R}h^{*L} - 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*} - 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*R} + 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*L} \\ + 2 \cdot h^{*R}hh^{*L} + 2 \cdot (h^{*R}hh^{*L})^{*} + 2 \cdot (h^{*R}hh^{*L})^{*R} + 2 \cdot (h^{*R}hh^{*L})^{*L} \end{array}}{\begin{array}{l} 7 \cdot (h^{*R}h)^{*L}h^{*R}h + ((h^{*R}h)^{*L}h^{*R})^{*}h + ((h^{*R}h)^{*L}h^{*R})^{*R}h - 5 \cdot ((h^{*R}h)^{*L}h^{*R})^{*L}h \\ + 3 \cdot (h^{*L}h)^{*R}h^{*L}h - 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*}h - 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*R}h + 3 \cdot ((h^{*L}h)^{*R}h^{*L})^{*L}h \\ + 2 \cdot h^{*R}hh^{*L}h + 2 \cdot (h^{*R}hh^{*L})^{*}h + 2 \cdot (h^{*R}hh^{*L})^{*R}h + 2 \cdot (h^{*R}hh^{*L})^{*L}h \end{array}} \right) \tag{A-35}$$

and,

$$\begin{aligned}
\lambda h^{-1} &= (h^{*L}h)^{*R}h^{*L} \\
&- (((h^{*L}h)^{*R}h^{*L}) + ((h^{*L}h)^{*R}h^{*L})^{*} + ((h^{*L}h)^{*R}h^{*L})^{*R} + ((h^{*L}h)^{*R}h^{*L})^{*L}) \cdot (2/12) \\
&- (((h^{*L}h)^{*R}h^{*L}) - ((h^{*L}h)^{*R}h^{*L})^{*} + ((h^{*L}h)^{*R}h^{*L})^{*R} - ((h^{*L}h)^{*R}h^{*L})^{*L}) \cdot (1/4) \\
&+ (((h^{*R}h)^{*L}h^{*R}) - ((h^{*R}h)^{*L}h^{*R})^{*} + ((h^{*R}h)^{*L}h^{*R})^{*R} - ((h^{*R}h)^{*L}h^{*R})^{*L}) \cdot (1/4) \\
&+ ((h^{*R}hh^{*L}) + (h^{*R}hh^{*L})^{*} + (h^{*R}hh^{*L})^{*R} + (h^{*R}hh^{*L})^{*L}) \cdot (2/12)
\end{aligned} \tag{A-36}$$

$$\begin{aligned}
\lambda h^{-1} &= (h^{*L}h)^{*R}h^{*L} \cdot \frac{7}{12} + ((h^{*L}h)^{*R}h^{*L})^{*} \cdot \frac{1}{12} - ((h^{*L}h)^{*R}h^{*L})^{*R} \cdot \frac{5}{12} + ((h^{*L}h)^{*R}h^{*L})^{*L} \cdot \frac{1}{12} \\
&+ (h^{*R}h)^{*L}h^{*R} \cdot \frac{3}{12} - ((h^{*R}h)^{*L}h^{*R})^{*} \cdot \frac{3}{12} + ((h^{*R}h)^{*L}h^{*R})^{*R} \cdot \frac{3}{12} - ((h^{*R}h)^{*L}h^{*R})^{*L} \cdot \frac{3}{12} \\
&+ h^{*R}hh^{*L} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*R} \cdot \frac{2}{12} + (h^{*R}hh^{*L})^{*L} \cdot \frac{2}{12}
\end{aligned} \tag{A-37}$$

$$\therefore \quad h^{-1} = \left(\frac{\begin{array}{l} 7 \cdot (h^{*L}h)^{*R}h^{*L} + ((h^{*L}h)^{*R}h^{*L})^{*} - 5 \cdot ((h^{*L}h)^{*R}h^{*L})^{*R} + ((h^{*L}h)^{*R}h^{*L})^{*L} \\ + 3 \cdot (h^{*R}h)^{*L}h^{*R} - 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*} + 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*R} - 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*L} \\ + 2 \cdot h^{*R}hh^{*L} + 2 \cdot (h^{*R}hh^{*L})^{*} + 2 \cdot (h^{*R}hh^{*L})^{*R} + 2 \cdot (h^{*R}hh^{*L})^{*L} \end{array}}{\begin{array}{l} 7 \cdot (h^{*L}h)^{*R}h^{*L}h + ((h^{*L}h)^{*R}h^{*L})^{*}h - 5 \cdot ((h^{*L}h)^{*R}h^{*L})^{*R}h + ((h^{*L}h)^{*R}h^{*L})^{*L}h \\ + 3 \cdot (h^{*R}h)^{*L}h^{*R}h - 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*}h + 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*R}h - 3 \cdot ((h^{*R}h)^{*L}h^{*R})^{*L}h \\ + 2 \cdot h^{*R}hh^{*L}h + 2 \cdot (h^{*R}hh^{*L})^{*}h + 2 \cdot (h^{*R}hh^{*L})^{*R}h + 2 \cdot (h^{*R}hh^{*L})^{*L}h \end{array}} \right) \tag{A-38}$$

Formulas (A-35) and (A-38) represent two of the many possible ways to write the inverse, h^{-1}. If we now take these final formulas to represent $h^{-1} = \eta(w)/\lambda$, then here, $\lambda = 12 \cdot d$, where d is the determinant of the matrix form of h, given also by the expression in (A-2). In a sense, this is just a way of doing matrix algebra using quaternions.

The λ is the quaternion expansion of the 4×4 real matrix determinant; or, in the final case, 12 times this determinant. The numerator, $\eta(w)$, is the adjoint matrix written out in quaternion format; or, 12 times this adjoint in our final case re-scaling of terms. We re-scale for convenience, to keep the intermediate fractions like $1/12$ out of the numerator.

For a given bilinear form,

$$h = A_1 B'_1 + A_2 B'_2 + \cdots + A_n B'_n \;=\; \sum A_k B'_k$$

the 12 terms in this numerator for (A-35) have the forms[15]

$$\begin{aligned}
+7\cdot\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR} &= +7\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A^*_k\cdot B'^*_j\cdot B'^*_i\cdot B'_k\\
+1\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)^* &= +1\cdot\sum\sum\sum A_k\cdot A^*_j\cdot A_i\cdot B'^*_k\cdot B'_i\cdot B'_j\\
+1\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)\mathrm{R} &= +1\cdot\sum\sum\sum A_k\cdot A^*_j\cdot A_i\cdot B'^*_j\cdot B'^*_i\cdot B'_k\\
-5\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)\mathrm{L} &= -5\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A^*_k\cdot B'^*_k\cdot B'_i\cdot B'_j\\
\\
+3\cdot\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL} &= +3\cdot\sum\sum\sum A^*_j\cdot A^*_i\cdot A_k\cdot B'^*_i\cdot B'_j\cdot B'^*_k\\
-3\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)^* &= -3\cdot\sum\sum\sum A^*_k\cdot A_i\cdot A_j\cdot B'_k\cdot B'^*_j\cdot B'_i\\
-3\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)\mathrm{R} &= -3\cdot\sum\sum\sum A^*_k\cdot A_i\cdot A_j\cdot B'^*_i\cdot B'_j\cdot B'^*_k\\
+3\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)\mathrm{L} &= +3\cdot\sum\sum\sum A^*_j\cdot A^*_i\cdot A_k\cdot B'_k\cdot B'^*_j\cdot B'_i\\
\\
+2\cdot\mathrm{HR{\cdot}H{\cdot}HL} &= +2\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A_k\cdot B'_i\cdot B'_j\cdot B'^*_k\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)^* &= +2\cdot\sum\sum\sum A^*_k\cdot A^*_j\cdot A_i\cdot B'_k\cdot B'^*_j\cdot B'^*_i\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)\mathrm{R} &= +2\cdot\sum\sum\sum A^*_k\cdot A^*_j\cdot A_i\cdot B'_i\cdot B'_j\cdot B'_k*\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)\mathrm{L} &= +2\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A_k\cdot B'_k\cdot B'^*_j\cdot B'^*_i
\end{aligned}\tag{A-39}$$

and, the 12 terms in this numerator for (A-38) have the forms;

$$\begin{aligned}
+7\cdot\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL} &= +7\cdot\sum\sum\sum A^*_j\cdot A^*_i\cdot A_k\cdot B'^*_i\cdot B'_j\cdot B'^*_k\\
+1\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)^* &= +1\cdot\sum\sum\sum A^*_k\cdot A_i\cdot A_j\cdot B'_k\cdot B'^*_j\cdot B'_i\\
-5\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)\mathrm{R} &= -5\cdot\sum\sum\sum A^*_k\cdot A_i\cdot A_j\cdot B'^*_i\cdot B'_j\cdot B'^*_k\\
+1\cdot\big(\big(\mathrm{HL{\cdot}H}\big)\mathrm{R{\cdot}HL}\big)\mathrm{L} &= +1\cdot\sum\sum\sum A^*_j\cdot A^*_i\cdot A_k\cdot B'_k\cdot B'^*_j\cdot B'_i\\
\\
+3\cdot\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR} &= +3\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A^*_k\cdot B'^*_j\cdot B'^*_i\cdot B'_k\\
-3\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)^* &= -3\cdot\sum\sum\sum A_k\cdot A^*_j\cdot A_i\cdot B'^*_k\cdot B'_i\cdot B'_j\\
+3\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)\mathrm{R} &= +3\cdot\sum\sum\sum A_k\cdot A^*_j\cdot A_i\cdot B'^*_j\cdot B'^*_i\cdot B'_k\\
-3\cdot\big(\big(\mathrm{HR{\cdot}H}\big)\mathrm{L{\cdot}HR}\big)\mathrm{L} &= -3\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A^*_k\cdot B'^*_k\cdot B'_i\cdot B'_j\\
\\
+2\cdot\mathrm{HR{\cdot}H{\cdot}HL} &= +2\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A_k\cdot B'_i\cdot B'_j\cdot B'^*_k\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)^* &= +2\cdot\sum\sum\sum A^*_k\cdot A^*_j\cdot A_i\cdot B'_k\cdot B'^*_j\cdot B'^*_i\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)\mathrm{R} &= +2\cdot\sum\sum\sum A^*_k\cdot A^*_j\cdot A_i\cdot B'_i\cdot B'_j\cdot B'_k*\\
+2\cdot\big(\mathrm{HR{\cdot}H{\cdot}HL}\big)\mathrm{L} &= +2\cdot\sum\sum\sum A^*_i\cdot A_j\cdot A_k\cdot B'_k\cdot B'^*_j\cdot B'^*_i
\end{aligned}\tag{A-40}$$

Here the indicies, i, j, k, range from 1 to n. If we can arrange these bi-cubic terms into $A^*_. A_. A^*_. B'^*_. B'_. B'^*_.$ form, then the construction for the denominator becomes, $A^*_. A_. A^*_. B'^*_. B'_. B'^*_. \cdot A_. B'_. = A^*_. A_. A^*_. A_. B'^*_. B'_. B'^*_. B'_.$, and the consecutive factors fall into convenient pairs, $(A^*_. A_.)(A^*_. A_.)(B'^*_. B'_.)(B'^*_. B'_.)$, which makes these non-abelian factors easier to re-arrange into scalar values using the two known forms, $|A_s|^2 = A^*_s A_s \in \mathbb{R}$, when the indices are the same, and $(A^*_r A_s) + (A^*_s A_r) \in \mathbb{R}$, when the indicies differ.

QUATERNION REPRESENTATION OF MATRIX ALGEBRA

Every real 4×4 matrix, $H \in M(\mathbb{R}, 4)$, can be written as an hexpe number $\in \mathbb{X}_n$,

$$H = [a_{uv}] = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{llll} h_0\mathbf{E} & + h_{R1}\mathbf{I_R} & + h_{R2}\mathbf{J_R} & + h_{R3}\mathbf{K_R} \\ & + h_{L1}\mathbf{I_L} & + h_{L2}\mathbf{J_L} & + h_{L3}\mathbf{K_L} \\ & + h_{M1}\mathbf{I_M} & + h_{M2}\mathbf{J_M} & + h_{M3}\mathbf{K_M} \\ & + h_{A1}\mathbf{I_A} & + h_{A2}\mathbf{J_A} & + h_{A3}\mathbf{K_A} \\ & + h_{Z1}\mathbf{I_Z} & + h_{Z2}\mathbf{J_Z} & + h_{Z3}\mathbf{K_Z} \end{array} \qquad \text{(A-41)}$$

where the $\boldsymbol{E}, \boldsymbol{I}_R, \boldsymbol{J}_R, \ldots, \boldsymbol{K}_Z$ are the basis of $\mathbb{X}_n$, the matrix coefficients are a_{uv}, and the hexpe coefficients h_s. In eqns (2.36)-(2.37) of [PJ2], these previously discussed coefficients are related and given by the equations;

$$\begin{array}{lcl}
a_{00} & = & + h_0 - h_{M1} - h_{M2} - h_{M3} \\
a_{10} & = & + h_{R1} + h_{L1} + h_{A1} - h_{Z1} \\
a_{20} & = & + h_{R2} + h_{L2} + h_{A2} - h_{Z2} \\
a_{30} & = & + h_{R3} + h_{L3} + h_{A3} - h_{Z3} \\
 & & \\
a_{01} & = & - h_{R1} - h_{L1} + h_{A1} - h_{Z1} \\
a_{11} & = & + h_0 - h_{M1} + h_{M2} + h_{M3} \\
a_{21} & = & + h_{R3} - h_{L3} - h_{A3} - h_{Z3} \\
a_{31} & = & - h_{R2} + h_{L2} - h_{A2} - h_{Z2} \\
 & & \\
a_{02} & = & - h_{R2} - h_{L2} + h_{A2} - h_{Z2} \\
a_{12} & = & - h_{R3} + h_{L3} - h_{A3} - h_{Z3} \\
a_{22} & = & + h_0 + h_{M1} - h_{M2} + h_{M3} \\
a_{32} & = & + h_{R1} - h_{L1} - h_{A1} - h_{Z1} \\
 & & \\
a_{03} & = & - h_{R3} - h_{L3} + h_{A3} - h_{Z3} \\
a_{13} & = & + h_{R2} - h_{L2} - h_{A2} - h_{Z2} \\
a_{23} & = & - h_{R1} + h_{L1} - h_{A1} - h_{Z1} \\
a_{33} & = & + h_0 + h_{M1} + h_{M2} - h_{M3}
\end{array}
\qquad
\begin{array}{lcl}
h_0 & = & (+ a_{00} + a_{11} + a_{22} + a_{33})/4 \\
 & & \\
h_{M1} & = & (- a_{00} - a_{11} + a_{22} + a_{33})/4 \\
h_{M2} & = & (- a_{00} + a_{11} - a_{22} + a_{33})/4 \\
h_{M3} & = & (- a_{00} + a_{11} + a_{22} - a_{33})/4 \\
 & & \\
h_{A1} & = & (+ a_{10} + a_{01} - a_{32} - a_{23})/4 \\
h_{A2} & = & (+ a_{20} - a_{31} + a_{02} - a_{13})/4 \\
h_{A3} & = & (+ a_{30} - a_{21} - a_{12} + a_{03})/4 \\
 & & \\
h_{Z1} & = & (- a_{10} - a_{01} - a_{32} - a_{23})/4 \\
h_{Z2} & = & (- a_{20} - a_{31} - a_{02} - a_{13})/4 \\
h_{Z3} & = & (- a_{30} - a_{21} - a_{12} - a_{03})/4 \\
 & & \\
h_{R1} & = & (+ a_{10} - a_{01} + a_{32} - a_{23})/4 \\
h_{R2} & = & (+ a_{20} - a_{31} - a_{02} + a_{13})/4 \\
h_{R3} & = & (+ a_{30} + a_{21} - a_{12} - a_{03})/4 \\
 & & \\
h_{L1} & = & (+ a_{10} - a_{01} - a_{32} + a_{23})/4 \\
h_{L2} & = & (+ a_{20} + a_{31} - a_{02} - a_{13})/4 \\
h_{L3} & = & (+ a_{30} - a_{21} + a_{12} - a_{03})/4
\end{array}
\qquad \text{(A-42)}$$

QUATERNION EXPANSIONS OF THE ADJOINT MATRIX

$$H^\dagger = \begin{array}{l}
(\;\; (+ 7 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 3 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;) \\
+ \; (+ 1 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} - 3 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*} \\
+ \; (+ 1 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} - 3 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*R} \\
+ \; (- 5 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 3 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*L} \;\;)/12
\end{array} \qquad \text{(A-43)}$$

$$= \begin{array}{l}
(\;\; (+ 7 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 3 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;) \\
+ \; (+ 1 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} - 3 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*} \\
+ \; (- 5 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} + 3 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*R} \\
+ \; (+ 1 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} - 3 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \;)^{*L} \;\;)/12
\end{array} \qquad \text{(A-44)}$$

QUATERNION EXPANSIONS OF THE MATRIX DETERMINANT

$$det(H) = \begin{array}{l}
(\;\; + 7 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} \cdot H + 3 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} \cdot H + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \cdot H \\
\;\; + 1 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*} \cdot H - 3 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*} \cdot H \\
\;\; + 1 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*R} \cdot H - 3 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*R} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*R} \cdot H \\
\;\; - 5 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*L} \cdot H + 3 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*L} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*L} \cdot H \;\;)/12
\end{array} \qquad \text{(A-45)}$$

$$= \begin{array}{l}
(\;\; + 7 \cdot (H^{*L} \cdot H)^{*R} \cdot H^{*L} \cdot H + 3 \cdot (H^{*R} \cdot H)^{*L} \cdot H^{*R} \cdot H + 2 \cdot H^{*R} \cdot H \cdot H^{*L} \cdot H \\
\;\; + 1 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*} \cdot H - 3 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*} \cdot H \\
\;\; - 5 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*R} \cdot H + 3 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*R} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*R} \cdot H \\
\;\; + 1 \cdot ((H^{*L} \cdot H)^{*R} \cdot H^{*L})^{*L} \cdot H - 3 \cdot ((H^{*R} \cdot H)^{*L} \cdot H^{*R})^{*L} \cdot H + 2 \cdot (H^{*R} \cdot H \cdot H^{*L})^{*L} \cdot H \;\;)/12
\end{array} \qquad \text{(A-46)}$$

For a given matrix, H, it's adjoint, $H^\dagger$, can be given a quaternion expansion in more than one way. Equations (A-43) and (A-44) illustrate the formulas for two such expansions, written using the four conjugated states; each of the four lines in the formula having an outer conjugation from one of the four states. The inner expression conjugated, is, however, different from one line to the next. But, the difference is only in the numerical coefficients that modify the common conjugated cubes appearing in the sum. If these numerical coefficients were all the same, and of the same sign, then we'd be combining the four conjugated states of the same hexpe number, consequently, the adjoint would collapse to a scalar value, according to the rule (A-19). Since the adjoint matrix is, in general, a non-scalar entity, we expect to have some variations in the sign and magnitude of these numerical coefficients. However, the best, most symmetrical, or most efficient construction, for the adjoint, in terms of quaternion expansions, is the subject of ongoing research. Two quaternion expansions for the determinant are shown in eqns (A-45) and (A-46).

In hexpe basis format, the matrix, H, it's adjoint, $H^\dagger$, and determinant, $det(H)$, are;

$$\begin{aligned} H = {} & h_0 + h_{M1}\boldsymbol{I}_M + h_{M2}\boldsymbol{J}_M + h_{M3}\boldsymbol{K}_M + h_{A1}\boldsymbol{I}_A + h_{A2}\boldsymbol{J}_A + h_{A3}\boldsymbol{K}_A + h_{Z1}\boldsymbol{I}_Z + h_{Z2}\boldsymbol{J}_Z + h_{Z3}\boldsymbol{K}_Z \\ & + h_{R1}\boldsymbol{I}_R + h_{R2}\boldsymbol{J}_R + h_{R3}\boldsymbol{K}_R + h_{L1}\boldsymbol{I}_L + h_{L2}\boldsymbol{J}_L + h_{L3}\boldsymbol{K}_L \end{aligned} \tag{A-47}$$

$$\begin{aligned} H^\dagger = {} & w_0 + w_{M1}\boldsymbol{I}_M + w_{M2}\boldsymbol{J}_M + w_{M3}\boldsymbol{K}_M + w_{A1}\boldsymbol{I}_A + w_{A2}\boldsymbol{J}_A + w_{A3}\boldsymbol{K}_A + w_{Z1}\boldsymbol{I}_Z + w_{Z2}\boldsymbol{J}_Z + w_{Z3}\boldsymbol{K}_Z \\ & + w_{R1}\boldsymbol{I}_R + w_{R2}\boldsymbol{J}_R + w_{R3}\boldsymbol{K}_R + w_{L1}\boldsymbol{I}_L + w_{L2}\boldsymbol{J}_L + w_{L3}\boldsymbol{K}_L \end{aligned} \tag{A-48}$$

$$d = det(H) = H^\dagger H = HH^\dagger = S(H^\dagger H) = S(HH^\dagger) \tag{A-49}$$

$$\begin{aligned} & = h_0 w_0 (1 \cdot 1) \\ & + h_{M1}w_{M1}(\boldsymbol{I}_M \cdot \boldsymbol{I}_M) + h_{M2}w_{M2}(\boldsymbol{J}_M \cdot \boldsymbol{J}_M) + h_{M3}w_{M3}(\boldsymbol{K}_M \cdot \boldsymbol{K}_M) \\ & + h_{A1}w_{A1}(\boldsymbol{I}_A \cdot \boldsymbol{I}_A) + h_{A2}w_{A2}(\boldsymbol{J}_A \cdot \boldsymbol{J}_A) + h_{A3}w_{A3}(\boldsymbol{K}_A \cdot \boldsymbol{K}_A) \\ & + h_{Z1}w_{Z1}(\boldsymbol{I}_Z \cdot \boldsymbol{I}_Z) + h_{Z2}w_{Z2}(\boldsymbol{J}_Z \cdot \boldsymbol{J}_Z) + h_{Z3}w_{Z3}(\boldsymbol{K}_Z \cdot \boldsymbol{K}_Z) \\ & + h_{R1}w_{R1}(\boldsymbol{I}_R \cdot \boldsymbol{I}_R) + h_{R2}w_{R2}(\boldsymbol{J}_R \cdot \boldsymbol{J}_R) + h_{R3}w_{R3}(\boldsymbol{K}_R \cdot \boldsymbol{K}_R) \\ & + h_{L1}w_{L1}(\boldsymbol{I}_L \cdot \boldsymbol{I}_L) + h_{L2}w_{L2}(\boldsymbol{J}_L \cdot \boldsymbol{J}_L) + h_{L3}w_{L3}(\boldsymbol{K}_L \cdot \boldsymbol{K}_L) \end{aligned}$$

$$\begin{aligned} & = h_0w_0 + h_{M1}w_{M1} + h_{M2}w_{M2} + h_{M3}w_{M3} + h_{A1}w_{A1} + h_{A2}w_{A2} + h_{A3}w_{A3} + h_{Z1}w_{Z1} + h_{Z2}w_{Z2} + h_{Z3}w_{Z3} \\ & - h_{R1}w_{R1} - h_{R2}w_{R2} - h_{R3}w_{R3} - h_{L1}w_{L1} - h_{L2}w_{L2} - h_{L3}w_{L3} \end{aligned} \tag{A-2}$$

The inverse is then, $H^{-1} = \frac{H^\dagger}{det(H)}$, and the determinant has the form reminding us of the dot product of vectors, with $\pm$ signs corresponding to the particular squares of the unit basis elements–the R-L anti-commuting quaternions having the $-$ sign, while the commuting M-A-Z hypercomplex numbers producing the $+$ sign. Looked at another way, we can " factor " a scalar, $\lambda \in \mathbb{R}$, into two 16-dimensional hypercomplex numbers, $\lambda = H^\dagger \cdot H;\ H^\dagger, H \in \mathbb{X}_n$.

360 CUBES. To find other quaternion expansions for the adjoint and determinant we need to consider alternative conjugated cubes. Now, while there are only 32 unique *conjugated squares*, there happen to be 360 unique *conjugated cubes*. So, there are many more cubes to choose from in the search for these expansions. The squares need no more than *three* conjugations to establish the unique set, but the cubes need *five* conjugations. At the least, a square is simply, hh, and at most, it's of the form, $(h^{*R}h^{*R})^{*R}$. The cube ranges from, hhh, to, $(h^{*R}(h^{*R}h^{*L})^{*L})^{*R}$. To see why we don't need more than five conjugations on the cube, let's try to add one more to the outside;

$$((h^{*R}(h^{*R}h^{*L})^{*L})^{*R})^{*R} = h^{*R}(h^{*R}h^{*L})^{*L} \tag{A-50}$$

$$((h^{*R}(h^{*R}h^{*L})^{*L})^{*R})^{*L} = (h^{*R}(h^{*R}h^{*L})^{*L})^{*} = (h^{*R}h^{*L})^{*R}h^{*L} \tag{A-51}$$

$$((h^{*R}(h^{*R}h^{*L})^{*L})^{*R})^{*} = (h^{*R}(h^{*R}h^{*L})^{*L})^{*L} \tag{A-52}$$

Adding an extra $(\cdot)^{*R}$ simply cancels the outer one that is already there, an extra $(\cdot)^{*L}$ combines to form the normal conjugate $(\cdot)^{*}$ which then simplifies the expression further, while an extra $(\cdot)^{*}$ converts the existing

$(\cdot)^{*R}$ into an $(\cdot)^{*L}$. So, which ever way we try, the five conjugated cube remains at five or even reduces to a lesser count of conjugations. Similarly, if we introduce an extra conjugation somewhere inside the expression, the result is either to convert existing conjugations into other forms, or reduce the form again. So, five conjugations are the maximum required to establish the unique set of cubes[16]. Now the four conjugated states can be labeled, _ N R L, for unconjugated, normal conjugate, right conjugate, and left conjugate. In RPN notation[17], the algebraic expression for, $(h^R h^L)^*$, can be written, HRHLxN. Here the x represents multiplication of the two previous terms on the stack, and the _ N R L letters indicate conjugations to be applied to the top element of the stack at that point in the computation—the underscore _ actually representing " blank ", i.e. no letter there, for the unconjugated state. This is a convenient notation for writing symbolic code to implement hexpe algorithms, that also happens to be convenient for counting the number of squares and cubes in a particular set.

A square always has to have two H letters and one x multiplication operation, e.g. HHx is hh. Then, up to three conjugations can be included: H.H.x. ← here the dots . show the locations in the expression where the four _ N R L letters may appear. With 3 locations and 4 letters we'd get $4 \times 4 \times 4 = 64$ squares. However, the normal conjugate N on the outside right dot . always converts the inside expression into another form that's already included in the count of unique squares; e.g. $(h^{*R}h)^* \to h^* h^{*L}$, so that, HRHxN → HNHLx. Then again, a left conjugate L on that outside dot . can always be converted into a right conjugate; e.g. $(h^{*L}h^*)^{*L} \to ((h^{*L}h^*)^*)^{*R} \to (hh^{*R})^{*R}$, so that, HLHNxL → HHRxR; therefore, we never really need the two conjugations N L on the outer dot location. This means the unique squares are only $4 \times 4 \times 2 = 32$ in number. Similarly, a cube must have three H letters and two x multiplication operations; e.g. HHHxx is hhh. There are now 5 locations, H.H.H.x.x. or H.H.x.H.x. , in each of two possible sequences, for us to choose from, to place our 4 conjugations _ N R L. Again, the outermost right dot . doesn't require the two letters N L, in either of these sequences; so, only 2 choices remain there–same for inner x; however, a blank _ appearing there, i.e. H.H.xH.x. effectively duplicates an H.H.H.xx. , so leaves only 1 choice, R, and this is therefore always H.H.xRH.x. . A less obvious double pair factor swap yields another 24 duplicates; shown in the table below:

$$
360 = 384 - 24 = 256 + 128 - 24 = \underset{\text{H.H.H.x.x.}}{4\cdot4\cdot4\cdot2\cdot2} + \underset{\text{H.H.x.H.x.}}{4\cdot4\cdot1\cdot4\cdot2} - 24
$$

HHNHxRx	=	HHNxRHx	=	$h(h^*h)^{*R}$	=	$(hh^*)^{*R}h$
HHNHxRxR	=	HHNxRHxR	=	$(h(h^*h)^{*R})^{*R}$	=	$((hh^*)^{*R}h)^{*R}$
HRHLHNxRx	=	HNHLxRHRx	=	$h^{*R}(h^{*L}h^*)^{*R}$	=	$(h^*h^{*L})^{*R}h^{*R}$
HLHRHxRx	=	HHRxRHLx	=	$h^{*L}(h^{*R}h)^{*R}$	=	$(hh^{*R})^{*R}h^{*L}$
HRHNHRxRx	=	HRHNxRHRx	=	$h^{*R}(h^*h^{*R})^{*R}$	=	$(h^{*R}h^*)^{*R}h^{*R}$
HNHHNxRxR	=	HNHxRHNxR	=	$(h^*(hh^*)^{*R})^{*R}$	=	$((h^*h)^{*R}h^*)^{*R}$
HLHHLxRx	=	HLHxRHLx	=	$h^{*L}(hh^{*L})^{*R}$	=	$(h^{*L}h)^{*R}h^{*L}$
HNHHNxRx	=	HNHxRHNx	=	$h^*(hh^*)^{*R}$	=	$(h^*h)^{*R}h^*$
HRHLHNxRxR	=	HNHLxRHRxR	=	$(h^{*R}(h^{*L}h^*)^{*R})^{*R}$	=	$((h^*h^{*L})^{*R}h^{*R})^{*R}$
HLHRHxRxR	=	HHRxRHLxR	=	$(h^{*L}(h^{*R}h)^{*R})^{*R}$	=	$((hh^{*R})^{*R}h^{*L})^{*R}$
HRHNHRxRxR	=	HRHNxRHRxR	=	$(h^{*R}(h^*h^{*R})^{*R})^{*R}$	=	$((h^{*R}h^*)^{*R}h^{*R})^{*R}$
HLHHLxRxR	=	HLHxRHLxR	=	$(h^{*L}(hh^{*L})^{*R})^{*R}$	=	$((h^{*L}h)^{*R}h^{*L})^{*R}$
HHNHLxRx	=	HLHNxRHx	=	$h(h^*h^{*L})^{*R}$	=	$(h^{*L}h^*)^{*R}h$
HHLHxRx	=	HHLxRHx	=	$h(h^{*L}h)^{*R}$	=	$(hh^{*L})^{*R}h$
HHNHLxRxR	=	HLHNxRHxR	=	$(h(h^*h^{*L})^{*R})^{*R}$	=	$((h^{*L}h^*)^{*R}h)^{*R}$
HHLHxRxR	=	HHLxRHxR	=	$(h(h^{*L}h)^{*R})^{*R}$	=	$((hh^{*L})^{*R}h)^{*R}$
HNHHRxRxR	=	HRHxRHNxR	=	$(h^*(hh^{*R})^{*R})^{*R}$	=	$((h^{*R}h)^{*R}h^*)^{*R}$
HNHHRxRx	=	HRHxRHNx	=	$h^*(hh^{*R})^{*R}$	=	$(h^{*R}h)^{*R}h^*$
HNHRHNxRxR	=	HNHRxRHNxR	=	$(h^*(h^{*R}h^*)^{*R})^{*R}$	=	$((h^*h^{*R})^{*R}h^*)^{*R}$
HNHRHNxRx	=	HNHRxRHNx	=	$h^*(h^{*R}h^*)^{*R}$	=	$(h^*h^{*R})^{*R}h^*$
HRHLHRxRx	=	HRHLxRHRx	=	$h^{*R}(h^{*L}h^{*R})^{*R}$	=	$(h^{*R}h^{*L})^{*R}h^{*R}$
HLHRHLxRx	=	HLHRxRHLx	=	$h^{*L}(h^{*R}h^{*L})^{*R}$	=	$(h^{*L}h^{*R})^{*R}h^{*L}$
HRHLHRxRxR	=	HRHLxRHRxR	=	$(h^{*R}(h^{*L}h^{*R})^{*R})^{*R}$	=	$((h^{*R}h^{*L})^{*R}h^{*R})^{*R}$
HLHRHLxRxR	=	HLHRxRHLxR	=	$(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}$	=	$((h^{*L}h^{*R})^{*R}h^{*L})^{*R}$

Table shows the non-obvious equivalent cubes; on the left in RPN notation, then again on the right in our usual symbol format. Two pairwise permutations of order equivalates, i.e. the normal conjugate obeys the rule, $(gh)^* = h^*g^*$, with the reversal of the pair of factors, but the partial conjugates do not, instead, two pairs must be reversed simultaneously, e.g. $(h^{*R}h)^{*R}\boldsymbol{h}^* \to \boldsymbol{h}^* \cdot (h^{*R}h)^{*R} \to h^*(\boldsymbol{h}^{*R}\boldsymbol{h})^{*R} \to h^*(\boldsymbol{h} \cdot \boldsymbol{h}^{*R})^{*R}$, to obtain equal expressions—only avilable in cubic and higher!

TABLE OF THE 360 CONJUGATED CUBES
AND THEIR EXTENSION TERMS
PAGE 1-OF-6: (1-60)

No.	CUBE	S	R	L	M	A	Z
0	$H^\dagger$	0	0	0	0	0	0
1	HHHxx	$0\bar{1}\bar{2}3\bar{4}5$	$01455\bar{6}\bar{6}\bar{8}99$	$01455\bar{6}\bar{6}\bar{8}99$	–	–	–
2	HHHxxR	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}67889\bar{9}$	$01455\bar{6}\bar{6}\bar{8}99$	–	–	–
3	HHHxRx	$011\bar{2}\bar{3}$	04558	03899	–	–	–
4	HHxRHx	$011\bar{2}\bar{3}$	04558	$0\bar{3}899$	–	–	–
5	HHHRxx	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{3}899$	–	–	–
6	HHRHxx	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{2}\bar{2}4899$	–	–	–
7	HRHHxx	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	03899	–	–	–
8	HNHNHNxxR	$0\bar{1}\bar{2}3\bar{4}5$	$01455\bar{6}\bar{6}\bar{8}99$	$0\bar{2}\bar{5}67889\bar{9}$	–	–	–
9	HHNHNxRx	$011\bar{3}\bar{4}$	03899	04558	–	–	–
10	HNHNxRHx	$011\bar{3}\bar{4}$	$0\bar{3}899$	04558	–	–	–
11	HHHLxx	$011\bar{3}\bar{4}$	$0\bar{3}899$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
12	HHLHxx	$011\bar{3}\bar{4}$	$0\bar{2}\bar{2}4899$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
13	HLHHxx	$011\bar{3}\bar{4}$	03899	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
14	HNHNHNxx	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}67889\bar{9}$	$0\bar{2}\bar{5}67889\bar{9}$	–	–	–
15	HHNHNxx	$0\bar{1}\bar{3}5$	$01\bar{3}\bar{6}\bar{6}8$	$01\bar{3}\bar{6}\bar{6}8$	–	–	–
16	HNHNHxx	$0\bar{1}\bar{3}5$	$013\bar{6}\bar{6}8$	$013\bar{6}\bar{6}8$	–	–	–
17	HHHNxx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$\bar{0}\bar{1}366\bar{8}$	–	–	–
18	HHNHxx	$0\bar{1}\bar{3}5$	$\bar{0}122\bar{4}66\bar{8}$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
19	HNHHxx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	–	–	–
20	HHHxRxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	03899	–	–	–
21	HHxRHxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{3}899$	–	–	–
22	HHHRxxR	$011\bar{2}\bar{3}$	04558	$0\bar{3}899$	–	–	–
23	HHRHxxR	$011\bar{2}\bar{3}$	0114558	$0\bar{2}\bar{2}4899$	–	–	–
24	HRHHxxR	$011\bar{2}\bar{3}$	04558	03899	–	–	–
25	HHNHNxRxR	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	04558	–	–	–
26	HNHNxRHxR	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	04558	–	–	–
27	HHHLxxR	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
28	HHLHxxR	$011\bar{3}\bar{4}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
29	HLHHxxR	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
30	HHNHNxxR	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$01\bar{3}\bar{6}\bar{6}8$	–	–	–
31	HNHNHxxR	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$013\bar{6}\bar{6}8$	–	–	–
32	HHHNxxR	$0\bar{1}\bar{3}5$	$01\bar{3}\bar{6}\bar{6}8$	$\bar{0}\bar{1}366\bar{8}$	–	–	–
33	HHNHxxR	$0\bar{1}\bar{3}5$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
34	HNHHxxR	$0\bar{1}\bar{3}5$	$013\bar{6}\bar{6}8$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	–	–	–
35	HHHRxRx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$01\bar{2}\bar{2}4899$	–	–	–
36	HHRHxRx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	$01\bar{3}899$	–	–	–
37	HRHHxRx	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}67889\bar{9}$	$01455\bar{6}\bar{6}\bar{8}99$	–	–	–
38	HHxRHRx	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}67889\bar{9}$	$01455\bar{6}\bar{6}\bar{8}99$	–	–	–
39	HHRxRHx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	013899	–	–	–
40	HRHxRHx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$01\bar{2}\bar{2}4899$	–	–	–
41	HHRHRxx	$0\bar{1}\bar{2}\bar{3}$	014558	013899	–	–	–
42	HRHHRxx	$0\bar{1}\bar{2}\bar{3}$	$0\bar{1}4558$	$01\bar{2}\bar{2}4899$	–	–	–
43	HRHRHxx	$0\bar{1}\bar{2}\bar{3}$	014558	$01\bar{3}899$	–	–	–
44	HLHNHNxxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
45	HNHLHNxxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	–	–	–
46	HNHNHLxxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
47	HHLHNxRx	$0\bar{1}$	$\bar{0}\bar{1}$	$01\bar{2}\bar{2}34$	–	–	–
48	HHNHLxRx	$0\bar{1}$	$\bar{0}1$	01	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$
49	HRHNHNxRx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$01\bar{3}\bar{6}\bar{6}8$	–	–	–
50	HNHNxRHRx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$013\bar{6}\bar{6}8$	–	–	–
51	HNHLxRHx	$0\bar{1}$	$\bar{0}\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	–	–	–
52	HHRHLxx	$0\bar{1}$	$\bar{0}1$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	–	–	–
53	HRHHLxx	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	–	–	–
54	HHLHRxx	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	$\bar{0}1$	–	–	–
55	HRHLHxx	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	$\bar{0}1$	–	–	–
56	HLHHRxx	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	–	–	–
57	HLHRHxx	$0\bar{1}$	$\bar{0}1$	$\bar{0}\bar{1}223\bar{4}$	–	–	–
58	HLHNHNxx	$011\bar{2}\bar{3}$	04558	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
59	HNHLHNxx	$011\bar{2}\bar{3}$	0114558	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	–	–	–
60	HNHNHLxx	$011\bar{2}\bar{3}$	04558	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–

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No.	CUBE	S	R	L	M	A	Z
61	HHLHNxx	0 1 1	$\bar{0}$	0	–	–	–
62	HHNHLxx	0 1 1	$\bar{0}\ \bar{1}\ \bar{1}$	$0\ \bar{2}\ \bar{2}\ 3\ 4$	–	–	–
63	HRHNHNxx	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$\bar{0}\ 3\ \bar{8}\ \bar{9}\ \bar{9}$	0 4 5 5 8	–	–	–
64	HNHNHRxx	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$\bar{0}\ \bar{3}\ \bar{8}\ \bar{9}\ \bar{9}$	0 4 5 5 8	–	–	–
65	HLHNHxx	0 1 1	$\bar{0}\ \bar{1}\ \bar{1}$	$0\ \bar{2}\ \bar{2}\ \bar{3}\ 4$	–	–	–
66	HNHLHxx	0 1 1	$\bar{0}$	0	–	–	–
67	HHRHNxx	0 1 1	0	$\bar{0}$	–	–	–
68	HRHHNxx	0 1 1	0 1 1	$\bar{0}\ 2\ 2\ \bar{3}\ \bar{4}$	–	–	–
69	HHNHRxx	0 1 1	$0\ \bar{2}\ \bar{2}\ 3\ 4$	$\bar{0}\ \bar{1}\ \bar{1}$	–	–	–
70	HRHNHxx	0 1 1	$0\ \bar{2}\ \bar{2}\ \bar{3}\ 4$	$\bar{0}\ \bar{1}\ \bar{1}$	–	–	–
71	HNHHRxx	0 1 1	0 1 1	$\bar{0}\ 2\ 2\ 3\ \bar{4}$	–	–	–
72	HNHRHxx	0 1 1	0	$\bar{0}$	–	–	–
73	HNHNxRHNxR	$0\ 1\ 1\ \bar{2}\ \bar{3}$	0 4 5 5 8	$\bar{0}\ \bar{3}\ \bar{8}\ \bar{9}\ \bar{9}$	–	–	–
74	HNHNHNxRxR	$0\ 1\ 1\ \bar{2}\ \bar{3}$	0 4 5 5 8	$\bar{0}\ 3\ \bar{8}\ 9\ \bar{9}$	–	–	–
75	HHxRHNxR	$0\ 1\ 1\ \bar{3}\ \bar{4}$	0 3 8 9 9	$\bar{0}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	–	–	–
76	HNHHxRxR	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$0\ \bar{3}\ 8\ 9\ 9$	$\bar{0}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	–	–	–
77	HRHNHNxxR	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$0\ \bar{3}\ 8\ 9\ 9$	0 4 5 5 8	–	–	–
78	HNHRHNxxR	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$0\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	0 1 1 4 5 5 8	–	–	–
79	HNHNHRxxR	$0\ 1\ 1\ \bar{3}\ \bar{4}$	0 3 8 9 9	0 4 5 5 8	–	–	–
80	HNHHNxxR	$0\ \bar{1}\ \bar{3}\ 5$	$\bar{0}\ 1\ 2\ 2\ \bar{4}\ 6\ 6\ \bar{8}$	$0\ \bar{1}\ \bar{2}\ \bar{2}\ 4\ \bar{6}\ \bar{6}\ 8$	–	–	–
81	HHHLxRx	$0\ \bar{1}$	$0\ 1\ \bar{2}\ \bar{2}\ 3\ 4$	$\bar{0}\ \bar{1}$	–	–	–
82	HHLHxRx	$0\ \bar{1}$	0 1	$\bar{0}\ 1$	$\bar{0}\ \bar{1}$	$\bar{0}\ \bar{1}$	$\bar{0}\ \bar{1}$
83	HLHHxRx	$0\ \bar{1}\ \bar{3}\ 5$	$0\ 1\ \bar{3}\ \bar{6}\ \bar{6}\ 8$	$\bar{0}\ \bar{1}\ \bar{3}\ 6\ 6\ \bar{8}$	–	–	–
84	HHxRHLx	$0\ \bar{1}\ \bar{3}\ 5$	$0\ 1\ 3\ \bar{6}\ \bar{6}\ 8$	$\bar{0}\ \bar{1}\ 3\ 6\ \bar{6}\ \bar{8}$	–	–	–
85	HLHxRHx	$0\ \bar{1}$	$0\ 1\ \bar{2}\ \bar{2}\ \bar{3}\ 4$	$\bar{0}\ \bar{1}$	–	–	–
86	HHRHNxRx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	$0\ 1\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	$\bar{0}\ \bar{1}\ 4\ \bar{5}\ 5\ \bar{8}$	–	–	–
87	HHNHRxRx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	$0\ 1\ \bar{3}\ 8\ 9\ 9$	$\bar{0}\ 1\ \bar{4}\ \bar{5}\ 5\ \bar{8}$	–	–	–
88	HLHNHNxRx	$0\ \bar{1}\ \bar{2}\ 3\ \bar{4}\ 5$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	$0\ \bar{2}\ \bar{5}\ 6\ 7\ 8\ 8\ \bar{9}$	–	–	–
89	HNHNxRHLx	$0\ \bar{1}\ \bar{2}\ 3\ \bar{4}\ 5$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	$0\ \bar{2}\ \bar{5}\ 6\ 7\ 8\ 8\ \bar{9}$	–	–	–
90	HRHNxRHx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	0 1 3 8 9 9	$\bar{0}\ 1\ \bar{4}\ \bar{5}\ 5\ \bar{8}$	–	–	–
91	HNHRxRHx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	$0\ 1\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	$\bar{0}\ \bar{1}\ 4\ \bar{5}\ 5\ \bar{8}$	–	–	–
92	HHLHLxx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	0 1 3 8 9 9	0 1 4 5 5 8	–	–	–
93	HLHHLxx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	$0\ 1\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	$0\ \bar{1}\ 4\ 5\ 5\ 8$	–	–	–
94	HLHLHxx	$0\ \bar{1}\ \bar{3}\ \bar{4}$	$0\ 1\ \bar{3}\ 8\ 9\ 9$	0 1 4 5 5 8	–	–	–
95	HNHRHNxx	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$\bar{0}\ 2\ 2\ \bar{4}\ 8\ \bar{9}\ \bar{9}$	0 1 1 4 5 5 8	–	–	–
96	HLHHNxx	0 1 1	$\bar{0}\ 2\ 2\ \bar{3}\ \bar{4}$	0 1 1	–	–	–
97	HNHHLxx	0 1 1	$\bar{0}\ 2\ 2\ 3\ \bar{4}$	0 1 1	–	–	–
98	HNHNxRHNx	$0\ 1\ 1\ \bar{2}\ \bar{3}$	$\bar{0}\ \bar{4}\ 5\ \bar{5}\ 8$	$\bar{0}\ 3\ \bar{8}\ 9\ \bar{9}$	–	–	–
99	HNHNHNxRx	$0\ 1\ 1\ \bar{2}\ \bar{3}$	$\bar{0}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	$\bar{0}\ 3\ \bar{8}\ \bar{9}\ \bar{9}$	–	–	–
100	HHxRHNx	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$\bar{0}\ 3\ \bar{8}\ \bar{9}\ \bar{9}$	$\bar{0}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	–	–	–
101	HNHHxRx	$0\ 1\ 1\ \bar{3}\ \bar{4}$	$\bar{0}\ 3\ 8\ \bar{9}\ \bar{9}$	$\bar{0}\ 4\ \bar{5}\ 5\ \bar{8}$	–	–	–
102	HNHHNxx	$0\ \bar{1}\ \bar{3}\ 5$	$0\ \bar{1}\ \bar{2}\ \bar{2}\ 4\ \bar{6}\ \bar{6}\ 8$	$0\ \bar{1}\ \bar{2}\ \bar{2}\ 4\ \bar{6}\ \bar{6}\ 8$	–	–	–
103	HHHNxRx	0 1 1	$\bar{0}\ 2\ 2\ \bar{3}\ \bar{4}$	$\bar{0}\ 2\ 2\ \bar{3}\ \bar{4}$	–	–	–
104	HHNHxRx	0 1 1	$\bar{0}\ \bar{1}\ \bar{1}$	$\bar{0}\ \bar{1}\ \bar{1}$	0 1 1	0 1 1	0 1 1
105	HNHxRHx	0 1 1	$\bar{0}\ 2\ 2\ 3\ \bar{4}$	$\bar{0}\ 2\ 2\ 3\ \bar{4}$	–	–	–
106	HHHRxRxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	0 1 4 5 5 8	$0\ 1\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	–	–	–
107	HHRHxRxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	$0\ \bar{1}\ 4\ 5\ 5\ 8$	$0\ 1\ \bar{3}\ 8\ 9\ 9$	–	–	–
108	HRHHxRxR	$0\ \bar{1}\ \bar{2}\ 3\ \bar{4}\ 5$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	–	–	–
109	HHxRHRxR	$0\ \bar{1}\ \bar{2}\ 3\ \bar{4}\ 5$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	$0\ 1\ 4\ 5\ 5\ \bar{6}\ \bar{6}\ \bar{8}\ 9\ 9$	–	–	–
110	HHRxRHxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	$0\ \bar{1}\ 4\ 5\ 5\ 8$	0 1 3 8 9 9	–	–	–
111	HRHxRHxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	0 1 4 5 5 8	$0\ 1\ \bar{2}\ \bar{2}\ 4\ 8\ 9\ 9$	–	–	–
112	HHRHRxxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	$\bar{0}\ \bar{1}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	0 1 3 8 9 9	–	–	–
113	HNHHNxRx	0 1 1	0 1 1	0 1 1	0 1 1	0 1 1	0 1 1
114	HRHRHxxR	$0\ \bar{1}\ \bar{2}\ \bar{3}$	$\bar{0}\ \bar{1}\ \bar{4}\ \bar{5}\ \bar{5}\ \bar{8}$	$0\ 1\ \bar{3}\ 8\ 9\ 9$	–	–	–
115	HHLHNxRxR	$0\ \bar{1}$	0 1	$0\ 1\ \bar{2}\ \bar{2}\ \bar{3}\ 4$	–	–	–
116	HHNHLxRxR	$0\ \bar{1}$	$0\ \bar{1}$	0 1	0 1	0 1	0 1
117	HRHNHNxRxR	$0\ \bar{1}\ \bar{3}\ 5$	$0\ 1\ 3\ \bar{6}\ \bar{6}\ 8$	$0\ 1\ \bar{3}\ \bar{6}\ \bar{6}\ 8$	–	–	–
118	HNHNxRHRxR	$0\ \bar{1}\ \bar{3}\ 5$	$0\ 1\ \bar{3}\ \bar{6}\ \bar{6}\ 8$	$0\ 1\ 3\ \bar{6}\ \bar{6}\ 8$	–	–	–
119	HNHLxRHxR	$0\ \bar{1}$	0 1	$0\ 1\ \bar{2}\ \bar{2}\ \bar{3}\ 4$	–	–	–
120	HHRHLxxR	$0\ \bar{1}$	$0\ \bar{1}$	$\bar{0}\ \bar{1}\ 2\ 2\ 3\ \bar{4}$	–	–	–

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No.	CUBE	S	R	L	M	A	Z
121	HRHHLxxR	$0\bar{1}$	01	$\bar{0}\bar{1}$	–	–	–
122	HHLHRxxR	$0\bar{1}$	$01\bar{2}\bar{2}34$	$\bar{0}1$	–	–	–
123	HRHLHxxR	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	$\bar{0}1$	–	–	–
124	HLHHRxxR	$0\bar{1}$	01	$\bar{0}\bar{1}$	–	–	–
125	HLHRHxxR	$0\bar{1}$	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	–	–	–
126	HHLHNxxR	011	0	0	–	–	–
127	HHNHLxxR	011	011	$0\bar{2}\bar{2}34$	–	–	–
128	HLHNHxxR	011	011	$0\bar{2}\bar{2}\bar{3}4$	–	–	–
129	HNHLHxxR	011	0	0	–	–	–
130	HHRHNxxR	011	$\bar{0}$	$\bar{0}$	–	–	–
131	HRHHNxxR	011	$\bar{0}\bar{1}\bar{1}$	$\bar{0}22\bar{3}\bar{4}$	–	–	–
132	HHNHRxxR	011	$\bar{0}22\bar{3}\bar{4}$	$\bar{0}\bar{1}\bar{1}$	–	–	–
133	HRHNHxxR	011	$\bar{0}223\bar{4}$	$\bar{0}\bar{1}\bar{1}$	–	–	–
134	HNHHRxxR	011	$\bar{0}\bar{1}\bar{1}$	$\bar{0}223\bar{4}$	–	–	–
135	HNHRHxxR	011	$\bar{0}$	$\bar{0}$	–	–	–
136	HHHLxRxR	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	$\bar{0}\bar{1}$	–	–	–
137	HHLHxRxR	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}1$	01	01	01
138	HLHHxRxR	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	–	–	–
139	HHxRHLxR	$0\bar{1}\bar{3}\bar{5}$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$\bar{0}\bar{1}366\bar{8}$	–	–	–
140	HLHxRHxR	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	$\bar{0}\bar{1}$	–	–	–
141	HHRHNxRxR	$0\bar{1}\bar{3}\bar{4}$	$025\bar{6}7\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
142	HHNHRxRxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$\bar{0}1\bar{4}\bar{5}5\bar{8}$	–	–	–
143	HLHNHNxRxR	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}6788\bar{9}$	$0\bar{2}\bar{5}6788\bar{9}$	–	–	–
144	HNHNxRHLxR	$0\bar{1}\bar{2}3\bar{4}5$	$0\bar{2}\bar{5}6788\bar{9}$	$0\bar{2}\bar{5}6788\bar{9}$	–	–	–
145	HRHNxRHxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	$\bar{0}1\bar{4}\bar{5}5\bar{8}$	–	–	–
146	HNHRxRHxR	$0\bar{1}\bar{3}\bar{4}$	$025\bar{6}7\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
147	HHLHLxxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	014558	–	–	–
148	HLHHLxxR	$0\bar{1}\bar{3}\bar{4}$	$025\bar{6}7\bar{9}$	$0\bar{1}4558$	–	–	–
149	HLHLHxxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	014558	–	–	–
150	HLHHNxxR	011	$0\bar{2}\bar{2}34$	011	–	–	–
151	HNHHLxxR	011	$0\bar{2}\bar{2}\bar{3}4$	011	–	–	–
152	HHHNxRxR	011	$0\bar{2}\bar{2}34$	$\bar{0}22\bar{3}\bar{4}$	–	–	–
153	HHNHxRxR	011	011	$\bar{0}\bar{1}\bar{1}$	$\bar{0}\bar{1}\bar{1}$	$\bar{0}\bar{1}\bar{1}$	$\bar{0}\bar{1}\bar{1}$
154	HNHxRHxR	011	$0\bar{2}\bar{2}\bar{3}4$	$\bar{0}223\bar{4}$	–	–	–
155	HHRHRxRx	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}\bar{8}99$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
156	HRHHRxRx	$011\bar{2}\bar{3}$	0114558	$0\bar{3}899$	–	–	–
157	HRHRHxRx	$011\bar{2}\bar{3}$	04558	$0\bar{2}\bar{2}4899$	–	–	–
158	HHRxRHRx	$011\bar{2}\bar{3}$	04558	$0\bar{2}\bar{2}4899$	–	–	–
159	HRHxRHRx	$011\bar{2}\bar{3}$	0114558	03899	–	–	–
160	HRHRxRHx	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}\bar{8}99$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
161	HRHRHRxx	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}\bar{5}\bar{5}6689\bar{9}\bar{9}$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
162	HLHLHNxxR	$0\bar{1}\bar{2}\bar{3}$	014558	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
163	HLHNHLxxR	$0\bar{1}\bar{2}\bar{3}$	$0\bar{1}4558$	$025\bar{6}7\bar{9}$	–	–	–
164	HNHLHLxxR	$0\bar{1}\bar{2}\bar{3}$	014558	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
165	HHLHLxRx	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
166	HRHLHNxRx	011	011	0	0	0	0
167	HRHNHLxRx	011	0	$0\bar{2}\bar{2}34$	–	–	–
168	HLHNxRHRx	011	0	$0\bar{2}\bar{2}\bar{3}4$	–	–	–
169	HLHLxRHx	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$03\bar{6}\bar{6}8$	–	–	–
170	HRHRHLxx	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$\bar{0}366\bar{8}$	–	–	–
171	HRHLHRxx	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
172	HLHRHRxx	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
173	HLHLHNxx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
174	HLHNHLxx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	$025\bar{6}7\bar{9}$	–	–	–
175	HNHLHLxx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
176	HRHLHNxx	$0\bar{1}$	$0\bar{1}$	$01\bar{2}\bar{2}34$	–	–	–
177	HRHNHLxx	$0\bar{1}$	01	01	–	–	–
178	HLHNHRxx	$0\bar{1}$	01	01	–	–	–
179	HNHLHRxx	$0\bar{1}$	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	–	–	–
180	HRHRHNxx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–

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No.	CUBE	S	R	L	M	A	Z
181	HRHNHRxx	$0\bar{1}\bar{3}\bar{4}$	$0256\bar{6}7\bar{9}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
182	HNHRHRxx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
183	HLHNxRHNxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$025\bar{6}7\bar{9}$	–	–	–
184	HNHLxRHNxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
185	HNHLHNxRxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	–	–	–
186	HNHNHLxRxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$025\bar{6}7\bar{9}$	–	–	–
187	HHRxRHNxR	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	–	–	–
188	HNHHRxRxR	$0\bar{1}$	$\bar{0}1$	$\bar{0}\bar{1}$	0 1	0 1	0 1
189	HNHRHxRxR	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	–	–	–
190	HRHLHNxxR	$0\bar{1}$	$\bar{0}1$	$01\bar{2}\bar{2}34$	–	–	–
191	HRHNHLxxR	$0\bar{1}$	$\bar{0}\bar{1}$	01	–	–	–
192	HLHRHNxxR	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	$0\bar{1}$	–	–	–
193	HNHRHLxxR	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	$0\bar{1}$	–	–	–
194	HLHNHRxxR	$0\bar{1}$	$\bar{0}\bar{1}$	01	–	–	–
195	HNHLHRxxR	$0\bar{1}$	$\bar{0}1$	$01\bar{2}\bar{2}\bar{3}4$	–	–	–
196	HHRHLxRx	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	$\bar{0}366\bar{8}$	–	–	–
197	HRHHLxRx	$011\bar{3}\bar{4}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
198	HHLHRxRx	$011\bar{3}5$	$\bar{0}366\bar{8}$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
199	HRHLHxRx	$011\bar{3}\bar{4}$	$\bar{0}\bar{3}\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
200	HLHHRxRx	011	$\bar{0}$	$\bar{0}223\bar{4}$	–	–	–
201	HLHRHxRx	011	$\bar{0}\bar{1}\bar{1}$	$\bar{0}$	0	0	0
202	HRHxRHLx	011	$\bar{0}$	$\bar{0}22\bar{3}\bar{4}$	–	–	–
203	HHLxRHRx	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
204	HRHLxRHx	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
205	HLHxRHRx	$011\bar{3}\bar{4}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
206	HLHRxRHx	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}2\bar{2}\bar{4}66\bar{8}$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
207	HRHRHNxRx	011	$\bar{0}223\bar{4}$	$\bar{0}$	–	–	–
208	HRHNHRxRx	011	$\bar{0}$	$\bar{0}\bar{1}\bar{1}$	0	0	0
209	HLHLHNxRx	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	–	–	–
210	HLHNHLxRx	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
211	HLHNxRHLx	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
212	HNHLxRHLx	$011\bar{2}\bar{3}$	$\bar{0}45\bar{5}\bar{8}$	$\bar{0}22\bar{4}89\bar{9}$	–	–	–
213	HNHRxRHRx	011	$\bar{0}22\bar{3}\bar{4}$	$\bar{0}$	–	–	–
214	HRHLHLxx	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
215	HLHRHLxx	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
216	HLHLHRxx	$011\bar{3}5$	$\bar{0}366\bar{8}$	$03\bar{6}\bar{6}8$	–	–	–
217	HLHRHNxx	$0\bar{1}$	$01\bar{2}\bar{2}34$	$0\bar{1}$	–	–	–
218	HNHRHLxx	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	$0\bar{1}$	–	–	–
219	HLHNxRHNx	$0\bar{1}\bar{2}\bar{3}$	014558	$025\bar{6}7\bar{9}$	–	–	–
220	HNHLxRHNx	$0\bar{1}\bar{2}\bar{3}$	$0\bar{1}4558$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
221	HNHLHNxRx	$0\bar{1}\bar{2}\bar{3}$	$0\bar{1}4558$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	–	–	–
222	HNHNHLxRx	$0\bar{1}\bar{2}\bar{3}$	014558	$025\bar{6}7\bar{9}$	–	–	–
223	HHRxRHNx	$0\bar{1}$	01	$\bar{0}\bar{1}22\bar{3}\bar{4}$	–	–	–
224	HNHHRxRx	$0\bar{1}$	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$
225	HNHRHxRx	$0\bar{1}$	01	$\bar{0}\bar{1}223\bar{4}$	–	–	–
226	HRHHNxRx	$0\bar{1}\bar{3}5$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}\bar{1}366\bar{8}$	–	–	–
227	HRHNHxRx	$0\bar{1}\bar{3}5$	$013\bar{6}\bar{6}8$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
228	HHNxRHRx	$0\bar{1}\bar{3}5$	$01\bar{3}\bar{6}\bar{6}8$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
229	HNHxRHRx	$0\bar{1}\bar{3}5$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	–	–	–
230	HRHNxRHNxR	$0\bar{1}$	$01\bar{2}\bar{2}34$	01	–	–	–
231	HNHRHNxRxR	$0\bar{1}$	01	$0\bar{1}$	0 1	0 1	0 1
232	HNHNHRxRxR	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	01	–	–	–
233	HHLxRHNxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{2}\bar{2}4899$	014558	–	–	–
234	HLHxRHNxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{3}899$	$0\bar{1}4558$	–	–	–
235	HNHHLxRxR	$0\bar{1}\bar{3}\bar{4}$	013899	$0\bar{1}4558$	–	–	–
236	HNHLHxRxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{2}\bar{2}4899$	014558	–	–	–
237	HRHRHNxxR	$0\bar{1}\bar{3}\bar{4}$	013899	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
238	HRHNHRxxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{2}\bar{2}4899$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
239	HNHRHRxxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{3}899$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
240	HHNxRHNxR	011	$\bar{0}22\bar{3}\bar{4}$	$0\bar{2}\bar{2}34$	–	–	–

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No.	CUBE	S	R	L	M	A	Z
241	HNHHNxRxR	011	$\bar{0}\bar{1}\bar{1}$	011	$\bar{0}\bar{1}\bar{1}$	$\bar{0}\bar{1}\bar{1}$	$\bar{0}\bar{1}\bar{1}$
242	HNHNHxRxR	011	$\bar{0}223\bar{4}$	$0\bar{2}\bar{2}\bar{3}4$	–	–	–
243	HLHHLxRx	011	0	011	0	0	0
244	HLHLHxRx	011	$0\bar{2}\bar{2}34$	0	–	–	–
245	HHLxRHLx	011	$0\bar{2}\bar{2}\bar{3}4$	0	–	–	–
246	HLHRHNxRx	$011\bar{3}\bar{4}$	$0\bar{3}899$	0114558	–	–	–
247	HLHNHRxRx	$011\bar{3}\bar{4}$	$0\bar{2}\bar{2}4899$	04558	–	–	–
248	HRHNxRHLx	$011\bar{3}\bar{4}$	$0\bar{2}\bar{2}4899$	04558	–	–	–
249	HNHRxRHLx	$011\bar{3}\bar{4}$	03899	0114558	–	–	–
250	HLHLHLxx	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}\bar{8}99$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	–	–	–
251	HRHNxRHNx	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	01	–	–	–
252	HNHRHNxRx	$0\bar{1}$	$\bar{0}\bar{1}$	$0\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$	$\bar{0}\bar{1}$
253	HNHNHRxRx	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	01	–	–	–
254	HHLxRHNx	$0\bar{1}\bar{3}\bar{4}$	$025\bar{6}7\bar{9}$	014558	–	–	–
255	HLHxRHNx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$0\bar{1}4558$	–	–	–
256	HNHHLxRx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$0\bar{1}4558$	–	–	–
257	HNHLHxRx	$0\bar{1}\bar{3}\bar{4}$	$025\bar{6}7\bar{9}$	014558	–	–	–
258	HLHHNxRx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
259	HLHNHxRx	$0\bar{1}\bar{3}5$	$\bar{0}122\bar{4}66\bar{8}$	$013\bar{6}\bar{6}8$	–	–	–
260	HHNxRHLx	$0\bar{1}\bar{3}5$	$\bar{0}122\bar{4}66\bar{8}$	$01\bar{3}\bar{6}\bar{6}8$	–	–	–
261	HNHxRHLx	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
262	HHNxRHNx	011	$0\bar{2}\bar{2}34$	$0\bar{2}\bar{2}34$	–	–	–
263	HRHHRxxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}1\bar{4}\bar{5}\bar{5}\bar{8}$	$01\bar{2}\bar{2}4899$	–	–	–
264	HNHNHxRx	011	$0\bar{2}\bar{2}\bar{3}4$	$0\bar{2}\bar{2}\bar{3}4$	–	–	–
265	HHRHRxRxR	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
266	HRHHRxRxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{3}899$	–	–	–
267	HRHRHxRxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{2}\bar{2}4899$	–	–	–
268	HHRxRHRxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	$0\bar{2}\bar{2}4899$	–	–	–
269	HRHxRHRxR	$011\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	03899	–	–	–
270	HRHRxRHxR	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
271	HRHRHRxxR	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}\bar{8}99$	$0455\bar{6}\bar{6}\bar{8}99$	–	–	–
272	HHLHLxRxR	$011\bar{3}5$	$\bar{0}366\bar{8}$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
273	HRHLHNxRxR	011	$\bar{0}\bar{1}\bar{1}$	0	$\bar{0}$	$\bar{0}$	$\bar{0}$
274	HRHNHLxRxR	011	$\bar{0}$	$0\bar{2}\bar{2}34$	–	–	–
275	HLHNxRHRxR	011	$\bar{0}$	$0\bar{2}\bar{2}\bar{3}4$	–	–	–
276	HLHLxRHxR	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$03\bar{6}\bar{6}8$	–	–	–
277	HRHRHLxxR	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$\bar{0}366\bar{8}$	–	–	–
278	HRHLHRxxR	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
279	HLHRHRxxR	$011\bar{3}5$	$\bar{0}366\bar{8}$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
280	HHRHLxRxR	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}366\bar{8}$	–	–	–
281	HRHHLxRxR	$011\bar{3}\bar{4}$	$0\bar{2}\bar{2}4899$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
282	HHLHRxRxR	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
283	HRHLHxRxR	$011\bar{3}\bar{4}$	03899	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
284	HLHHRxRxR	011	0	$\bar{0}223\bar{4}$	–	–	–
285	HLHRHxRxR	011	011	$\bar{0}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
286	HRHxRHLxR	011	0	$\bar{0}22\bar{3}\bar{4}$	–	–	–
287	HHLxRHRxR	$011\bar{3}\bar{4}$	$0\bar{3}899$	$\bar{0}\bar{1}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
288	HRHLxRHxR	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	–	–	–
289	HLHxRHRxR	$011\bar{3}\bar{4}$	$0\bar{2}\bar{2}4899$	$\bar{0}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
290	HLHRxRHxR	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
291	HRHRHNxRxR	011	$0\bar{2}\bar{2}\bar{3}4$	$\bar{0}$	–	–	–
292	HRHNHRxRxR	011	0	$\bar{0}\bar{1}\bar{1}$	$\bar{0}$	$\bar{0}$	$\bar{0}$
293	HLHLHNxRxR	$011\bar{2}\bar{3}$	04558	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	–	–	–
294	HLHNHLxRxR	$011\bar{2}\bar{3}$	0114558	$\bar{0}\bar{3}\bar{8}\bar{9}\bar{9}$	–	–	–
295	HLHNxRHLxR	$011\bar{2}\bar{3}$	0114558	$\bar{0}3\bar{8}\bar{9}\bar{9}$	–	–	–
296	HNHLxRHLxR	$011\bar{2}\bar{3}$	04558	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	–	–	–
297	HNHRxRHRxR	011	$0\bar{2}\bar{2}34$	$\bar{0}$	–	–	–
298	HRHLHLxxR	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
299	HLHRHLxxR	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
300	HLHLHRxxR	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$03\bar{6}\bar{6}8$	–	–	–

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No.	CUBE	S	R	L	M	A	Z
301	HRHHNxRxR	$0\bar{1}\bar{3}5$	$\bar{0}122\bar{4}66\bar{8}$	$\bar{0}\bar{1}366\bar{8}$	–	–	–
302	HRHNHxRxR	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
303	HHNxRHRxR	$0\bar{1}\bar{3}5$	$\bar{0}\bar{1}366\bar{8}$	$\bar{0}122\bar{4}66\bar{8}$	–	–	–
304	HNHxRHRxR	$0\bar{1}\bar{3}5$	$\bar{0}122\bar{4}66\bar{8}$	$\bar{0}\bar{1}\bar{3}66\bar{8}$	–	–	–
305	HLHHLxRxR	011	$\bar{0}$	011	$\bar{0}$	$\bar{0}$	$\bar{0}$
306	HLHLHxRxR	011	$\bar{0}22\bar{3}\bar{4}$	0	–	–	–
307	HHLxRHLxR	011	$\bar{0}223\bar{4}$	0	–	–	–
308	HLHRHNxRxR	$011\bar{3}\bar{4}$	$\bar{0}3\bar{8}\bar{9}\bar{9}$	0114558	–	–	–
309	HLHNHRxRxR	$011\bar{3}\bar{4}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	04558	–	–	–
310	HRHNxRHLxR	$011\bar{3}\bar{4}$	$\bar{0}22\bar{4}\bar{8}\bar{9}\bar{9}$	04558	–	–	–
311	HNHRxRHLxR	$011\bar{3}\bar{4}$	$\bar{0}\bar{3}\bar{8}\bar{9}\bar{9}$	0114558	–	–	–
312	HLHLHLxxR	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	–	–	–
313	HLHHNxRxR	$0\bar{1}\bar{3}5$	$01\bar{3}\bar{6}\bar{6}8$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
314	HLHNHxRxR	$0\bar{1}\bar{3}5$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	$013\bar{6}\bar{6}8$	–	–	–
315	HHNxRHLxR	$0\bar{1}\bar{3}5$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	$01\bar{3}\bar{6}\bar{6}8$	–	–	–
316	HNHxRHLxR	$0\bar{1}\bar{3}5$	$01\bar{3}\bar{6}\bar{6}8$	$0\bar{1}\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
317	HRHRHRxRx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	013899	–	–	–
318	HRHRxRHRx	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$01\bar{3}899$	–	–	–
319	HRHLHLxRx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	014558	–	–	–
320	HLHLxRHRx	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	014558	–	–	–
321	HLHLxRHNxR	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}899$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	–	–	–
322	HNHLHLxRxR	$011\bar{2}3\bar{4}5$	$0455\bar{6}\bar{6}899$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	–	–	–
323	HRHRxRHNxR	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$\bar{0}366\bar{8}$	–	–	–
324	HNHRHRxRxR	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
325	HRHRHLxRx	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	–	–	–
326	HRHLHRxRx	$0\bar{1}$	$0\bar{1}$	$\bar{0}1$	$\bar{0}1$	$\bar{0}1$	$\bar{0}1$
327	HLHRHRxRx	$0\bar{1}\bar{3}\bar{4}$	013899	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
328	HRHRxRHLx	$0\bar{1}\bar{3}\bar{4}$	$01\bar{3}899$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
329	HLHRxRHRx	$0\bar{1}$	$01\bar{2}\bar{2}34$	$\bar{0}\bar{1}223\bar{4}$	–	–	–
330	HLHLHLxRx	$0\bar{1}\bar{2}\bar{3}$	014558	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
331	HLHLxRHLx	$0\bar{1}\bar{2}\bar{3}$	014558	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
332	HLHLxRHNx	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	$\bar{0}\bar{4}\bar{5}\bar{5}668\bar{9}\bar{9}$	–	–	–
333	HNHLHLxRx	$011\bar{2}3\bar{4}5$	$\bar{0}\bar{4}5\bar{5}668\bar{9}\bar{9}$	$\bar{0}\bar{4}5\bar{5}668\bar{9}\bar{9}$	–	–	–
334	HRHRxRHNx	$011\bar{3}5$	$\bar{0}366\bar{8}$	$\bar{0}366\bar{8}$	–	–	–
335	HNHRHRxRx	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$\bar{0}\bar{3}66\bar{8}$	–	–	–
336	HRHLxRHNxR	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
337	HLHRxRHNxR	$011\bar{3}5$	$\bar{0}366\bar{8}$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
338	HNHRHLxRxR	$011\bar{3}5$	$\bar{0}\bar{3}66\bar{8}$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
339	HNHLHRxRxR	$011\bar{3}5$	$\bar{0}\bar{1}\bar{1}22\bar{4}66\bar{8}$	$03\bar{6}\bar{6}8$	–	–	–
340	HLHRHLxRx	$0\bar{1}$	$\bar{0}1$	$0\bar{1}$	$\bar{0}1$	$\bar{0}1$	$\bar{0}1$
341	HLHLHRxRx	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	$01\bar{2}\bar{2}\bar{3}4$	–	–	–
342	HRHLxRHLx	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	$01\bar{2}\bar{2}34$	–	–	–
343	HRHLxRHNx	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$0\bar{3}\bar{6}\bar{6}8$	–	–	–
344	HLHRxRHNx	$011\bar{3}5$	$0\bar{3}\bar{6}\bar{6}8$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
345	HNHRHLxRx	$011\bar{3}5$	$03\bar{6}\bar{6}8$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	–	–	–
346	HNHLHRxRx	$011\bar{3}5$	$011\bar{2}\bar{2}4\bar{6}\bar{6}8$	$03\bar{6}\bar{6}8$	–	–	–
347	HRHRHRxRxR	$0\bar{1}\bar{2}\bar{3}$	014558	013899	–	–	–
348	HRHRxRHRxR	$0\bar{1}\bar{2}\bar{3}$	014558	$01\bar{3}899$	–	–	–
349	HRHLHLxRxR	$0\bar{1}\bar{3}\bar{4}$	$01\bar{3}899$	014558	–	–	–
350	HLHLxRHRxR	$0\bar{1}\bar{3}\bar{4}$	013899	014558	–	–	–
351	HRHRHLxRxR	$0\bar{1}$	$\bar{0}\bar{1}223\bar{4}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	–	–	–
352	HRHLHRxRxR	$0\bar{1}$	$\bar{0}1$	$\bar{0}1$	$0\bar{1}$	$0\bar{1}$	$0\bar{1}$
353	HLHRHRxRxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
354	HRHRxRHLxR	$0\bar{1}\bar{3}\bar{4}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	–	–	–
355	HLHRHLxRxR	$0\bar{1}$	$0\bar{1}$	$0\bar{1}$	$0\bar{1}$	$0\bar{1}$	$0\bar{1}$
356	HLHLHLxRxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}3\bar{8}\bar{9}\bar{9}$	–	–	–
357	HLHLxRHLxR	$0\bar{1}\bar{2}\bar{3}$	$\bar{0}\bar{1}\bar{4}\bar{5}\bar{5}\bar{8}$	$\bar{0}\bar{1}\bar{3}\bar{8}\bar{9}\bar{9}$	–	–	–
358	HLHRxRHRxR	$0\bar{1}$	$\bar{0}\bar{1}22\bar{3}\bar{4}$	$\bar{0}\bar{1}223\bar{4}$	–	–	–
359	HLHLHRxRxR	$0\bar{1}$	$01\bar{2}\bar{2}34$	$01\bar{2}\bar{2}\bar{3}4$	–	–	–
360	HRHLxRHLxR	$0\bar{1}$	$01\bar{2}\bar{2}\bar{3}4$	$01\bar{2}\bar{2}34$	–	–	–

EXPLANATION OF THE TABLE: The first column just labels the cube with a number $1\cdots 360$, with no.0 being the adjoint $H^\dagger$, which is a combination of cubes. The second column shows the formula for the cube in RPN notation. The remaining columns give the scalar component, S, and the five R-L-M-A-Z vector sub-parts of the 15-dim vector part of the hexpe number for the cube. The components shown are given in terms of the weights, w_k, and the differences from those weights. The weights are the components of the adjoint, and are represented by a 0 in each column. The differences from these weights are represented by other numbers $1, 2, \ldots, 9$, which are just labels again that identify particular expression terms to be included. Now, although there are 360 unique cubes, many of the scalar components, S, of these cubes, are the same. In fact, all the scalar components have extension terms that are themselves constructed from only one to five distinct terms. So, we identify these parts and give them the labels $1, 2, 3, 4, 5$. These must be added to the adjoint weight, 0, to construct the component for the cube. If one of these expressions is to be subtracted from the weight instead, then a minus sign is shown in the table just above the numeric label. So, $\bar{2}$ means subtract the term with label 2. The differences are originally obtained by adding or subtracting the adjoint from the cube, then if subtracting the adjoint produces the smallest extension term we represent the cube component with, $+0+\cdots$, but if adding the adjoint produces the smallest extension term, we then represent the cube component with, $-0+\cdots$, instead. Consequently, some 0s have a minus bar above, $\bar{0}$, to indicate the weight is to be included with a minus sign prefix. Not all components can be reduced to simple terms, so most of the M-A-Z components are left out of the table, and a dash – appears in their location instead. The right and left hand quaternion components are all given, however, and need upto 9 sub-terms in our particular scheme–the scalar component would actually require 6 sub-terms if we followed the 'add or subtract the adjoint and choose the least' method, but by subtracting the adjoint only we reduce the count there to 5. There is a little ambiguity in this method, since measuring the length of an extension term is dependent on whether we count AAA + AAA as two terms or one 2.AAA term. Generally, we employ the latter. The extension term for each component on a cube is made up of several expression terms, but in the table shown, each of the R-L-M-A-Z columns represent the 3-vector sub-part of the cube, and so represent a group of three components with their three extension terms simultaneously. This organization is possible because these R-L-M-A-Z vary as 5 distinct parts, and not 15 distinct parts, in the alternation of conjugations producing the set of cubes.

Using this table we may then construct the various possible unique quaternion expansions for the adjoint and determinant. It is not necessary to know the definitions of the parts of the extension terms, but these are given in another table below for reference–All we need is the numeric labels and the signs for these terms, which are given above.

The 360 cubes are paritioned into exactly 10 *unequal* sets by scalar component alone. The 10 different possible values of the scalar, $S(\cdot)$, of a conjugated cube, are;

$$
\begin{array}{lclcrcl}
S_0 & = & 0\,\bar{1} & = & +0-1 & = & +w_0 - s_1 \\
S_1 & = & 0\,1\,1 & = & +0+1+1 & = & +w_0 + 2s_1 \\
S_2 & = & 0\,\bar{1}\,\bar{2}\,\bar{3} & = & +0-1-2-3 & = & +w_0 - s_1 - s_2 - s_3 \\
S_3 & = & 0\,1\,1\,\bar{2}\,\bar{3} & = & +0+1+1-2-3 & = & +w_0 + 2s_1 - s_2 - s_3 \\
S_4 & = & 0\,\bar{1}\,\bar{3}\,\bar{4} & = & +0-1-3-4 & = & +w_0 - s_1 - s_3 - s_4 \\
S_5 & = & 0\,1\,1\,\bar{3}\,\bar{4} & = & +0+1+1-3-4 & = & +w_0 + 2s_1 - s_3 - s_4 \\
S_6 & = & 0\,\bar{1}\,\bar{3}\,5 & = & +0-1-3+5 & = & +w_0 - s_1 - s_3 + s_5 \\
S_7 & = & 0\,1\,1\,\bar{3}\,5 & = & +0+1+1-3+5 & = & +w_0 + 2s_1 - s_3 + s_5 \\
S_8 & = & 0\,\bar{1}\,\bar{2}\,3\,\bar{4}\,5 & = & +0-1-2+3-4+5 & = & +w_0 - s_1 - s_2 + s_3 - s_4 + s_5 \\
S_9 & = & 0\,1\,1\,\bar{2}\,3\,\bar{4}\,5 & = & +0+1+1-2+3-4+5 & = & +w_0 + 2s_1 - s_2 + s_3 - s_4 + s_5
\end{array}
$$

The extension term, on each of the 10 scalar values, $S_j,\ j = 0, 1, .., 9$, is constructed from a particular combination of the five part-terms, $s_k,\ k = 1, 2, 3, 4, 5$. In this indexing scheme, the weight term can be thought of as the zeroth part-term, i.e. $s_0 = w_0$; a similar label and index scheme then follows for the R-L-M-A-Z, where r_0 is the 3-vector weight term, $r_0 = w_{R1}\boldsymbol{i} + w_{R2}\boldsymbol{j} + w_{R3}\boldsymbol{k}$, etc..In this way, we can translate the codes in the table of cubes into corresponding arithmetic expressions representing the component or vector sub-part of the hexpe number. The goal is to extract the s_0, or r_0 etc.., term, by making suitable combinations of cubes. For example, our previous construction (A-29) combined the triple conjugated cube $(h^{*R}h)^{*L}h^{*R}$, which has form, $S_1 = w_0 + 2s_1$, with the double conjugated cube $h^{*R}hh^{*L}$, with form, $S_0 = w_0 - s_1$, to find, $w_0 = s_0 = (2S_0 + S_1)/3$. But, any two such cubes, taken one from S_0 and one from S_1, can be used to construct w_0. In fact, our triple conjugated cube, $(h^{*R}h)^{*L}h^{*R}$, which has RPN form, HRHXLHRX, is not included directly in the cube table. It has an L just to the right of the inside X, and in our chosen symbol scheme only R can appear there. This cube is equivelent to another cube, $((h^{*R}h)^*)^{*R}h^{*R} = (h^*h^{*L})^{*R}h^{*R}$, which is a quadconjugated cube with RPN form, HNHLXRHRX, which is, again, not included either, because it is in that group of 24 that by double pair factor swap have duplicates that are already in the table. The cube is ultimately equivalent to $h^{*R}(h^{*L}h^*)^{*R}$, shown in the table of 24 duplicates on page[44] in the 3rd row, this has RPN form HRHLHNXRX, and appears as no.166 in the cube table. All the factor swap cubes with double H before X, i.e. H.H.X.H.X. are discarded in favor of their duplicates containing triple H before X, i.e. H.H.H.X.X. , in this table.

TABLE OF EXTENSION TERMS:

$$
\begin{aligned}
r_0 &= w_{R1}\boldsymbol{I}_R + w_{R2}\boldsymbol{J}_R + w_{R3}\boldsymbol{K}_R \\
r_1 &= \boldsymbol{I}_R \cdot 4 \cdot (+h_{L1}h_{M2}h_{M3} + h_{L2}h_{A1}h_{A2} + h_{L3}h_{Z1}h_{Z3} - h_{L1}h_{A1}h_{Z1} - h_{L2}h_{M3}h_{Z3} - h_{L3}h_{M2}h_{A2}) \\
&\quad + \boldsymbol{J}_R \cdot 4 \cdot (+h_{L1}h_{Z1}h_{Z2} + h_{L2}h_{M1}h_{M3} + h_{L3}h_{A2}h_{A3} - h_{L1}h_{M3}h_{A3} - h_{L2}h_{A2}h_{Z2} - h_{L3}h_{M1}h_{Z1}) \\
&\quad + \boldsymbol{K}_R \cdot 4 \cdot (+h_{L1}h_{A1}h_{A3} + h_{L2}h_{Z2}h_{Z3} + h_{L3}h_{M1}h_{M2} - h_{L1}h_{M2}h_{Z2} - h_{L2}h_{M1}h_{A1} - h_{L3}h_{A3}h_{Z3}) \\
r_2 &= \boldsymbol{I}_R \cdot 2 \cdot (+h_{R1}h_{M2}h_{M2} + h_{R1}h_{M3}h_{M3} + h_{R1}h_{A1}h_{A1} + h_{R1}h_{A2}h_{A2} + h_{R1}h_{Z1}h_{Z1} + h_{R1}h_{Z3}h_{Z3}) \\
&\quad + \boldsymbol{J}_R \cdot 2 \cdot (+h_{R2}h_{M1}h_{M1} + h_{R2}h_{M3}h_{M3} + h_{R2}h_{A2}h_{A2} + h_{R2}h_{A3}h_{A3} + h_{R2}h_{Z1}h_{Z1} + h_{R2}h_{Z2}h_{Z2}) \\
&\quad + \boldsymbol{K}_R \cdot 2 \cdot (+h_{R3}h_{M1}h_{M1} + h_{R3}h_{M2}h_{M2} + h_{R3}h_{A1}h_{A1} + h_{R3}h_{A3}h_{A3} + h_{R3}h_{Z2}h_{Z2} + h_{R3}h_{Z3}h_{Z3}) \\
r_3 &= \boldsymbol{I}_R \cdot 4 \cdot (+h_{R3}h_{L1}h_{Z3} + h_{R3}h_{L2}h_{M2} + h_{R3}h_{L3}h_{A1} - h_{R2}h_{L1}h_{A2} - h_{R2}h_{L2}h_{Z1} - h_{R2}h_{L3}h_{M3}) \\
&\quad + \boldsymbol{J}_R \cdot 4 \cdot (+h_{R1}h_{L1}h_{A2} + h_{R1}h_{L2}h_{Z1} + h_{R1}h_{L3}h_{M3} - h_{R3}h_{L1}h_{M1} - h_{R3}h_{L2}h_{A3} - h_{R3}h_{L3}h_{Z2}) \\
&\quad + \boldsymbol{K}_R \cdot 4 \cdot (+h_{R2}h_{L1}h_{M1} + h_{R2}h_{L2}h_{A3} + h_{R2}h_{L3}h_{Z2} - h_{R1}h_{L1}h_{Z3} - h_{R1}h_{L2}h_{M2} - h_{R1}h_{L3}h_{A1}) \\
r_4 &= \boldsymbol{I}_R \cdot 4 \cdot (+h_{R2}h_{M1}h_{Z3} + h_{R2}h_{M2}h_{A3} + h_{R2}h_{A1}h_{Z2} + h_{R3}h_{M1}h_{A2} + h_{R3}h_{M3}h_{Z2} + h_{R3}h_{A3}h_{Z1}) \\
&\quad + \boldsymbol{J}_R \cdot 4 \cdot (+h_{R1}h_{M1}h_{Z3} + h_{R1}h_{M2}h_{A3} + h_{R1}h_{A1}h_{Z2} + h_{R3}h_{M2}h_{Z1} + h_{R3}h_{M3}h_{A1} + h_{R3}h_{A2}h_{Z3}) \\
&\quad + \boldsymbol{K}_R \cdot 4 \cdot (+h_{R1}h_{M1}h_{A2} + h_{R1}h_{M3}h_{Z2} + h_{R1}h_{A3}h_{Z1} + h_{R2}h_{M2}h_{Z1} + h_{R2}h_{M3}h_{A1} + h_{R2}h_{A2}h_{Z3}) \\
r_5 &= \boldsymbol{I}_R \cdot 2 \cdot (+h_{R1}h_{M1}h_{M1} + h_{R1}h_{A3}h_{A3} + h_{R1}h_{Z2}h_{Z2}) \\
&\quad + \boldsymbol{J}_R \cdot 2 \cdot (+h_{R2}h_{M2}h_{M2} + h_{R2}h_{A1}h_{A1} + h_{R2}h_{Z3}h_{Z3}) \\
&\quad + \boldsymbol{K}_R \cdot 2 \cdot (+h_{R3}h_{M3}h_{M3} + h_{R3}h_{A2}h_{A2} + h_{R3}h_{Z1}h_{Z1}) \\
r_6 &= \boldsymbol{I}_R \cdot 2 \cdot (+h_{R1}h_{L1}h_{L1} + h_{R1}h_{L2}h_{L2} + h_{R1}h_{L3}h_{L3}) \\
&\quad + \boldsymbol{J}_R \cdot 2 \cdot (+h_{R2}h_{L1}h_{L1} + h_{R2}h_{L2}h_{L2} + h_{R2}h_{L3}h_{L3}) \\
&\quad + \boldsymbol{K}_R \cdot 2 \cdot (+h_{R3}h_{L1}h_{L1} + h_{R3}h_{L2}h_{L2} + h_{R3}h_{L3}h_{L3}) \\
r_7 &= \boldsymbol{I}_R \cdot 2 \cdot (+h_{R1}h_{R1}h_{R1} + h_{R1}h_{R2}h_{R2} + h_{R1}h_{R3}h_{R3}) \\
&\quad + \boldsymbol{J}_R \cdot 2 \cdot (+h_{R1}h_{R1}h_{R2} + h_{R2}h_{R2}h_{R2} + h_{R2}h_{R3}h_{R3}) \\
&\quad + \boldsymbol{K}_R \cdot 2 \cdot (+h_{R1}h_{R1}h_{R3} + h_{R2}h_{R2}h_{R3} + h_{R3}h_{R3}h_{R3}) \\
r_8 &= \boldsymbol{I}_R \cdot 4 \cdot (+h_0h_{L1}h_{M1} + h_0h_{L2}h_{A3} + h_0h_{L3}h_{Z2}) \\
&\quad + \boldsymbol{J}_R \cdot 4 \cdot (+h_0h_{L1}h_{Z3} + h_0h_{L2}h_{M2} + h_0h_{L3}h_{A1}) \\
&\quad + \boldsymbol{K}_R \cdot 4 \cdot (+h_0h_{L1}h_{A2} + h_0h_{L2}h_{Z1} + h_0h_{L3}h_{M3}) \\
r_9 &= \boldsymbol{I}_R \cdot 2 \cdot (+h_0h_0h_{R1}) + \boldsymbol{J}_R \cdot 2 \cdot (+h_0h_0h_{R2}) + \boldsymbol{K}_R \cdot 2 \cdot (+h_0h_0h_{R3})
\end{aligned}
$$

$$
\begin{aligned}
l_0 &= w_{L1}\boldsymbol{I}_L + w_{L2}\boldsymbol{J}_L + w_{L3}\boldsymbol{K}_L \\
l_1 &= \boldsymbol{I}_L \cdot 4 \cdot (+h_{R1}h_{M2}h_{M3} + h_{R2}h_{Z1}h_{Z2} + h_{R3}h_{A1}h_{A3} - h_{R1}h_{A1}h_{Z1} - h_{R2}h_{M3}h_{A3} - h_{R3}h_{M2}h_{Z2}) \\
&\quad + \boldsymbol{J}_L \cdot 4 \cdot (+h_{R1}h_{A1}h_{A2} + h_{R2}h_{M1}h_{M3} + h_{R3}h_{Z2}h_{Z3} - h_{R1}h_{M3}h_{Z3} - h_{R2}h_{A2}h_{Z2} - h_{R3}h_{M1}h_{A1}) \\
&\quad + \boldsymbol{K}_L \cdot 4 \cdot (+h_{R1}h_{Z1}h_{Z3} + h_{R2}h_{A2}h_{A3} + h_{R3}h_{M1}h_{M2} - h_{R1}h_{M2}h_{A2} - h_{R2}h_{M1}h_{Z1} - h_{R3}h_{A3}h_{Z3}) \\
l_2 &= \boldsymbol{I}_L \cdot 2 \cdot (+h_{L1}h_{M2}h_{M2} + h_{L1}h_{M3}J + h_{L1}h_{A1}h_{A1} + h_{L1}h_{A3}h_{A3} + h_{L1}h_{Z1}h_{Z1} + h_{L1}h_{Z2}h_{Z2}) \\
&\quad + \boldsymbol{J}_L \cdot 2 \cdot (+h_{L2}h_{M1}h_{M1} + h_{L2}h_{M3}J + h_{L2}h_{A1}h_{A1} + h_{L2}h_{A2}h_{A2} + h_{L2}h_{Z2}h_{Z2} + h_{L2}h_{Z3}h_{Z3}) \\
&\quad + \boldsymbol{K}_L \cdot 2 \cdot (+h_{L3}h_{M1}h_{M1} + h_{L3}h_{M2}h_{M2} + h_{L3}h_{A2}h_{A2} + h_{L3}h_{A3}h_{A3} + h_{L3}h_{Z1}h_{Z1} + h_{L3}h_{Z3}h_{Z3}) \\
l_3 &= \boldsymbol{I}_L \cdot 4 \cdot (+h_{R1}h_{L2}h_{Z2} + h_{R2}h_{L2}h_{A1} + h_{R3}h_{L2}h_{M3} - h_{R1}h_{L3}h_{A3} - h_{R2}h_{L3}h_{M2} - h_{R3}h_{L3}h_{Z1}) \\
&\quad + \boldsymbol{J}_L \cdot 4 \cdot (+h_{R1}h_{L3}h_{M1} + h_{R2}h_{L3}h_{Z3} + h_{R3}h_{L3}h_{A2} - h_{R1}h_{L1}h_{Z2} - h_{R2}h_{L1}h_{A1} - h_{R3}h_{L1}h_{M3}) \\
&\quad + \boldsymbol{K}_L \cdot 4 \cdot (+h_{R1}h_{L1}h_{A3} + h_{R2}h_{L1}h_{M2} + h_{R3}h_{L1}h_{Z1} - h_{R1}h_{L2}h_{M1} - h_{R2}h_{L2}h_{Z3} - h_{R3}h_{L2}h_{A2}) \\
l_4 &= \boldsymbol{I}_L \cdot 4 \cdot (+h_{L2}h_{M1}h_{A3} + h_{L2}h_{M2}h_{Z3} + h_{L2}h_{A2}h_{Z1} + h_{L3}h_{M1}h_{Z2} + h_{L3}h_{M3}h_{A2} + h_{L3}h_{A1}h_{Z3}) \\
&\quad + \boldsymbol{J}_L \cdot 4 \cdot (+h_{L1}h_{M1}h_{A3} + h_{L1}h_{M2}h_{Z3} + h_{L1}h_{A2}h_{Z1} + h_{L3}h_{M2}h_{A1} + h_{L3}h_{M3}h_{Z1} + h_{L3}h_{A3}h_{Z2}) \\
&\quad + \boldsymbol{K}_L \cdot 4 \cdot (+h_{L1}h_{M1}h_{Z2} + h_{L1}h_{M3}h_{A2} + h_{L1}h_{A1}h_{Z3} + h_{L2}h_{M2}h_{A1} + h_{L2}h_{M3}h_{Z1} + h_{L2}h_{A3}h_{Z2}) \\
l_5 &= \boldsymbol{I}_L \cdot 2 \cdot (+h_{L1}h_{M1}h_{M1} + h_{L1}h_{A2}h_{A2} + h_{L1}h_{Z3}h_{Z3}) \\
&\quad + \boldsymbol{J}_L \cdot 2 \cdot (+h_{L2}h_{M2}h_{M2} + h_{L2}h_{A3}h_{A3} + h_{L2}h_{Z1}h_{Z1}) \\
&\quad + \boldsymbol{K}_L \cdot 2 \cdot (+h_{L3}h_{M3}J + h_{L3}h_{A1}h_{A1} + h_{L3}h_{Z2}h_{Z2}) \\
l_6 &= \boldsymbol{I}_L \cdot 2 \cdot (+h_{R1}h_{R1}h_{L1} + h_{R2}h_{R2}h_{L1} + h_{R3}h_{R3}h_{L1}) \\
&\quad + \boldsymbol{J}_L \cdot 2 \cdot (+h_{R1}h_{R1}h_{L2} + h_{R2}h_{R2}h_{L2} + h_{R3}h_{R3}h_{L2}) \\
&\quad + \boldsymbol{K}_L \cdot 2 \cdot (+h_{R1}h_{R1}h_{L3} + h_{R2}h_{R2}h_{L3} + h_{R3}h_{R3}h_{L3}) \\
l_7 &= \boldsymbol{I}_L \cdot 2 \cdot (+h_{L1}h_{L1}h_{L1} + h_{L1}h_{L2}h_{L2} + h_{L1}h_{L3}h_{L3}) \\
&\quad + \boldsymbol{J}_L \cdot 2 \cdot (+h_{L1}h_{L1}h_{L2} + h_{L2}h_{L2}h_{L2} + h_{L2}h_{L3}h_{L3}) \\
&\quad + \boldsymbol{K}_L \cdot 2 \cdot (+h_{L1}h_{L1}h_{L3} + h_{L2}h_{L2}h_{L3} + h_{L3}h_{L3}h_{L3}) \\
l_8 &= \boldsymbol{I}_L \cdot 4 \cdot (+h_0h_{R1}h_{M1} + h_0h_{R2}h_{Z3} + h_0h_{R3}h_{A2}) \\
&\quad + \boldsymbol{J}_L \cdot 4 \cdot (+h_0h_{R1}h_{A3} + h_0h_{R2}h_{M2} + h_0h_{R3}h_{Z1}) \\
&\quad + \boldsymbol{K}_L \cdot 4 \cdot (+h_0h_{R1}h_{Z2} + h_0h_{R2}h_{A1} + h_0h_{R3}h_{M3}) \\
l_9 &= \boldsymbol{I}_L \cdot 2 \cdot (+h_0h_0h_{L1}) + \boldsymbol{J}_L \cdot 2 \cdot (+h_0h_0h_{L2}) + \boldsymbol{K}_L \cdot 2 \cdot (+h_0h_0h_{L3})
\end{aligned}
$$

TABLE OF EXTENSION TERMS: cont'd

$$
\begin{aligned}
s_0 &= w_0 \\
s_1 &= +4\cdot(h_{M1}h_{M2}h_{M3} + h_{A1}h_{A2}h_{A3} + h_{Z1}h_{Z2}h_{Z3} - h_{M1}h_{A1}h_{Z1} - h_{M2}h_{A2}h_{Z2} - h_{M3}h_{A3}h_{Z3}) \\
s_2 &= +4\cdot(h_0h_{L1}h_{L1} + h_0h_{L2}h_{L2} + h_0h_{L3}h_{L3}) \\
s_3 &= +4\cdot(h_{R1}h_{L1}h_{M1} + h_{R1}h_{L2}h_{A3} + h_{R1}h_{L3}h_{Z2} + h_{R2}h_{L1}h_{Z3} + h_{R2}h_{L2}h_{M2} + h_{R2}h_{L3}h_{A1} + h_{R3}h_{L1}h_{A2} + h_{R3}h_{L2}h_{Z1} + h_{R3}h_{L3}h_{M3}) \\
s_4 &= +4\cdot(h_0h_{R1}h_{R1} + h_0h_{R2}h_{R2} + h_0h_{R3}h_{R3}) \\
s_5 &= +4\cdot(h_0h_{M1}h_{M1} + h_0h_{M2}h_{M2} + h_0h_{M3}h_{M3} + h_0h_{A1}h_{A1} + h_0h_{A2}h_{A2} + h_0h_{A3}h_{A3} + h_0h_{Z1}h_{Z1} + h_0h_{Z2}h_{Z2} + h_0h_{Z3}h_{Z3}) \\
s_6 &= +2\cdot(h_0h_0h_0)
\end{aligned}
$$

$$
\begin{aligned}
m_0 &= w_{M1}\boldsymbol{I}_M + w_{M2}\boldsymbol{J}_M + w_{M3}\boldsymbol{K}_M \\
m_1 &= \boldsymbol{I}_M\cdot 4\cdot(+h_0h_{M2}h_{M3} + h_{R2}h_{L3}h_{Z1} + h_{R3}h_{L2}h_{A1} - h_0h_{A1}h_{Z1} - h_{R2}h_{L2}h_{M3} - h_{R3}h_{L3}h_{M2}) \\
&\quad + \boldsymbol{J}_M\cdot 4\cdot(+h_0h_{M1}h_{M3} + h_{R1}h_{L3}h_{A2} + h_{R3}h_{L1}h_{Z2} - h_0h_{A2}h_{Z2} - h_{R1}h_{L1}h_{M3} - h_{R3}h_{L3}h_{M1}) \\
&\quad + \boldsymbol{K}_M\cdot 4\cdot(+h_0h_{M1}h_{M2} + h_{R1}h_{L2}h_{Z3} + h_{R2}h_{L1}h_{A3} - h_0h_{A3}h_{Z3} - h_{R1}h_{L1}h_{M2} - h_{R2}h_{L2}h_{M1}) \\
a_0 &= w_{A1}\boldsymbol{I}_A + w_{A2}\boldsymbol{J}_A + w_{A3}\boldsymbol{K}_A \\
a_1 &= \boldsymbol{I}_A\cdot 4\cdot(+h_0h_{A2}h_{A3} + h_{R1}h_{L1}h_{Z1} + h_{R3}h_{L2}h_{M1} - h_0h_{M1}h_{Z1} - h_{R1}h_{L2}h_{A2} - h_{R3}h_{L1}h_{A3}) \\
&\quad + \boldsymbol{J}_A\cdot 4\cdot(+h_0h_{A1}h_{A3} + h_{R1}h_{L3}h_{M2} + h_{R2}h_{L2}h_{Z2} - h_0h_{M2}h_{Z2} - h_{R1}h_{L2}h_{A1} - h_{R2}h_{L3}h_{A3}) \\
&\quad + \boldsymbol{K}_A\cdot 4\cdot(+h_0h_{A1}h_{A2} + h_{R2}h_{L1}h_{M3} + h_{R3}h_{L3}h_{Z3} - h_0h_{M3}h_{Z3} - h_{R2}h_{L3}h_{A2} - h_{R3}h_{L1}h_{A1}) \\
z_0 &= w_{Z1}\boldsymbol{I}_Z + w_{Z2}\boldsymbol{J}_Z + w_{Z3}\boldsymbol{K}_Z \\
z_1 &= \boldsymbol{I}_Z\cdot 4\cdot(+h_0h_{Z2}h_{Z3} + h_{R1}h_{L1}h_{A1} + h_{R2}h_{L3}h_{M1} - h_0h_{M1}h_{A1} - h_{R1}h_{L3}h_{Z3} - h_{R2}h_{L1}h_{Z2}) \\
&\quad + \boldsymbol{J}_Z\cdot 4\cdot(+h_0h_{Z1}h_{Z3} + h_{R2}h_{L2}h_{A2} + h_{R3}h_{L1}h_{M2} - h_0h_{M2}h_{A2} - h_{R2}h_{L1}h_{Z1} - h_{R3}h_{L2}h_{Z3}) \\
&\quad + \boldsymbol{K}_Z\cdot 4\cdot(+h_0h_{Z1}h_{Z2} + h_{R1}h_{L2}h_{M3} + h_{R3}h_{L3}h_{A3} - h_0h_{M3}h_{A3} - h_{R1}h_{L3}h_{Z1} - h_{R3}h_{L2}h_{Z2})
\end{aligned}
$$

CORRESPONDING CUBES: A pair of cubes are called *corresponding cubes* if one is obtained from the other by flipping the hand of all the conjugates. These two cubes must have similar extension term part sequences in their opposite hand vectors. For example, HHHxxR and HHHxxL , are direct corresponding cubes. The latter does not exist directly in the table, but is equivalent to HNHNHNxxR , which *is* in the table. So, HHHxxR and HNHNHNxxR , are indirect corresponding cubes. These appear as no.2 and no.8, respectively, in the cube table. The R-COLUMN entry for the former, then, must be the same label sequence as the L-COLUMN entry for the latter; and, visa versa, the R-COLUMN entry for the latter must also be the same label sequence as the L-COLUMN entry for the former. It is necessary to properly align the definitions of the extension terms so that the labels attach to corresponding parts on the R and L coefficients for this observation to become manifest.

No.	CUBE	S	R	L	M	A	Z
2	HHHxxR	$0\,\bar{1}\,\bar{2}\,3\,\bar{4}\,5$	$0\,\bar{2}\,\bar{5}\,6\,7\,8\,8\,\bar{9}$	$0\,1\,4\,5\,5\,\bar{6}\,\bar{6}\,\bar{8}\,9\,9$	–	–	–
8	HNHNHNxxR	$0\,\bar{1}\,\bar{2}\,3\,\bar{4}\,5$	$0\,1\,4\,5\,5\,\bar{6}\,\bar{6}\,\bar{8}\,9\,9$	$0\,\bar{2}\,\bar{5}\,6\,7\,8\,8\,\bar{9}$	–	–	–
166	HRHLHNxRx	0 1 1	0 1 1	0	0	0	0
243	HLHHLxRx	0 1 1	0	0 1 1	0	0	0
113	HNHHNxRx	0 1 1	0 1 1	0 1 1	0 1 1	0 1 1	0 1 1
355	HLHRHLxRxR	$0\,\bar{1}$	$0\,\bar{1}$	$0\,\bar{1}$	$0\,\bar{1}$	$0\,\bar{1}$	$0\,\bar{1}$

We can then use this to double check the entries in the table. These two sequences are $0\,\bar{2}\,\bar{5}\,6\,7\,8\,8\,\bar{9}$ and $0\,1\,4\,5\,5\,\bar{6}\,\bar{6}\,\bar{8}\,9\,9$, respectively. Notice, also, that our two initially selected cubes, $(h^{*R}h)^{*L}h^{*R}$ and $(h^{*L}h)^{*R}h^{*L}$, defined in C-1 and C-2, on which we constructed our quaternion expansions, are corresponding cubes. In RPN these are HRHxLHRx and HLHxRHLx, which, in the table, are represented by the indirect corresponding pair, HRHLHNxRx and HLHHLxRx, with cubes no.166 and no.243. These cubes tend to have complementary entries in their component parts that facilitate the construction of quaternion expansions. In 360, are 136 pairs corresponding, and 88 self-corresponding cubes.

CONVERSIONS OF CORRESPONDING CUBES

$\longrightarrow$

HHHxxR	HHHxxL	$(hhh)^{*L}$	$((hhh)^{*})^{*R}$	$(h^{*}h^{*}h^{*})^{*R}$	HNHNHNxxR
HNHNHNxxR	HNHNHNxxL	$(h^{*}h^{*}h^{*})^{*L}$	$((h^{*}h^{*}h^{*})^{*})^{*R}$	$(hhh)^{*R}$	HHHxxR
HRHLHNxRx	HLHRHNxLx	$h^{*L}(h^{*R}h^{*})^{*L}$	$h^{*L}((h^{*R}h^{*})^{*})^{*R}$	$h^{*L}(hh^{*L})^{*R}$	HLHHLxRx
HLHHLxRx	HRHHRxLx	$h^{*R}(hh^{*R})^{*L}$	$h^{*R}((hh^{*R})^{*})^{*R}$	$h^{*R}(h^{*L}h^{*})^{*R}$	HRHLHNxRx
HNHHNxRx	HNHHNxLx	$h^{*}(hh^{*})^{*L}$	$h^{*}((hh^{*})^{*})^{*R}$	$h^{*}(hh^{*})^{*R}$	HNHHNxRx
HLHRHLxRxR	HRHLHRxLxL	$(h^{*R}(h^{*L}h^{*R})^{*L})^{*L}$	$((h^{*R}(h^{*L}h^{*R})^{*L})^{*})^{*R}$	$((h^{*L}h^{*R})^{*R}h^{*L})^{*R}$	HLHRxRHLxR

From the cube table, we find two cubes that are even more intersting than our previous selections. Cubes no.113 and no.355 have simple extensions that are easily made to vanish by combining twice the latter with one times the former; i.e. $3 \cdot H^\dagger = C113 + 2 \cdot C355$. Looking at the conversions table, we see that these are both self-corresponding cubes: HNHHNxRx → HNHHNxRx, under conversion, and, HLHRHLxRxR → HLHRxRHLxR. But, from the last row of the table of 24 factor swap duplicates on page[44], we see this latter is just the duplicate of HLHRHLxRxR once again. So, the C355 converts back to itself. We can then write the adjoint matrix as the sum of 2 cubes, instead of 12 cubes!

$$H^\dagger = \frac{1}{3} \cdot H^*(HH^*)^{*R} + \frac{2}{3} \cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R} \quad \text{(A-53)}$$

$$det(H) = \frac{1}{3} \cdot H^*(HH^*)^{*R}H + \frac{2}{3} \cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R}H \quad \text{(A-54)}$$

Here we combine a triconjugated cube with a pentaconjugated cube. If we thought of the normal conjugate as two conjugations, $(\cdot)^* = ((\cdot)^{*L})^{*R}$, then, this solution can be interpreted as the combination of two pentaconjugated cubes, in partial conjugates, instead.

$$H^\dagger = \frac{1}{3} \cdot (H^{*L})^{*R}(H(H^{*L})^{*R})^{*R} + \frac{2}{3} \cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R} \quad \text{(A-55)}$$

$$det(H) = \frac{1}{3} \cdot (H^{*L})^{*R}(H(H^{*L})^{*R})^{*R}H + \frac{2}{3} \cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R}H \quad \text{(A-56)}$$

At any rate, *five* is the maximum number of conjugations on a cube; so, it is interesting that the simplest solution found to date, combining the fewest terms possible, involves the maximal conjugate. At first glance, it might seem odd that there are three R-conjugates, but only two L-conjugates, in these cubes. Why should the right-hand be more prominant than the left? However, some R's can be exchanged for an L. For example, $(h^{*L}(h^{*R}h^{*L})^{*R})^{*R} = ((h^{*L}(h^{*R}h^{*L})^{*R})^*)^{*L} = ((h^{*R}h^{*L})^{*L}h^{*R})^{*L}$, so now we have three L's and two R's. Again, $(h^{*L}(h^{*R}h^{*L})^{*R})^{*R} = (h^{*L}((h^{*R}h^{*L})^*)^{*L})^{*R} = (h^{*L}(h^{*R}h^{*L})^{*L})^{*R}$, producing three L's and two R's. The pentaconjugated cube therefore has two different ways to convert to triple left. It is just our established convention to always select the duplicate cube with the right conjugate, i.e "xR" selected over "xL", that the formulas appear this way. The triple conjugated cube has one way to convert to triple left in it's pseudo-pentaconjugation form, since, $h^*(hh^*)^{*R} = h^*((hh^*)^*)^L = h^*(hh^*)^{*L}$, so we can write, $(h^{*L})^{*R}(h(h^{*L})^{*R})^{*R} = (h^{*L})^{*R}(h(h^{*L})^{*R})^{*L}$, also. With, $h = \sum A_i B'_i$, the two new cubes are found to already have the required form, $A^*_. A_. A^*_. B^*_. B_. B^*_.$, which we seek;

$$\begin{aligned} h^*(hh^*)^{*R} &= \left(\sum A_i B'_i\right)^* \left(\left(\sum A_j B'_j\right)\left(\sum A_k B'_k\right)^*\right)^{*R} = \left(\sum A^*_i B'^*_i\right)\left(\left(\sum A_j B'_j\right)\left(\sum A^*_k B'^*_k\right)\right)^{*R} \\ &= \left(\sum A^*_i B'^*_i\right)\left(\sum\sum A_j A^*_k B'_j B'^*_k\right)^{*R} = \left(\sum A^*_i B'^*_i\right)\left(\sum\sum A_k A^*_j B'_j B'^*_k\right) \\ &= \sum\sum\sum A^*_i A_k A^*_j B'^*_i B'_j B'^*_k \end{aligned} \quad \text{(A-57)}$$

$$\begin{aligned} 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R} &= 2\left(\left(\sum A_i B'_i\right)^{*L}\left(\left(\sum A_j B'_j\right)^{*R}\left(\sum A_k B'_k\right)^{*L}\right)^{*R}\right)^{*R} \\ &= 2\left(\left(\sum A_i B'^*_i\right)\left(\left(\sum A^*_j B'_j\right)\left(\sum A_k B'^*_k\right)\right)^{*R}\right)^{*R} \\ &= 2\left(\left(\sum A_i B'^*_i\right)\left(\sum\sum A^*_j A_k B'_j B'^*_k\right)^{*R}\right)^{*R} \\ &= 2\left(\left(\sum A_i B'^*_i\right)\left(\sum\sum A^*_k A_j B'_j B'^*_k\right)\right)^{*R} \\ &= 2\left(\sum\sum\sum A_i A^*_k A_j B'^*_i B'_j B'^*_k\right)^{*R} \\ &= 2\sum\sum\sum (A_i A^*_k A_j)^* B'^*_i B'_j B'^*_k \\ &= 2\sum\sum\sum A^*_j A_k A^*_i B'^*_i B'_j B'^*_k \end{aligned} \quad \text{(A-58)}$$

and no further re-arrangement of parameters is required to put the numerator into this convenient form. The maximum number of $1 \cdot A^*_. A_. A^*_. B^*_. B_. B^*_.$ terms, ever required, is n^3; see eqn (127). The minimum number of distinct numerator terms with general scalar coefficients, for arbitrary n, is $(2n^3 - 3n^2 + 4n)/3$, see eqn (144). In this final format, we may also now replace the $\sum\sum\sum$ with a single $\sum$, where the sum is understood to be over the three subscript indicies, i, j, k, all ranging from 1 to n. In the text of this paper, we discuss how the quartic factor, $B'^*_i B'_j B'^*_k B'_l$, appearing next, can be converted into the pure right hand form, $B_l B^*_k B_j B^*_i$, in the formula for the determinant, see eqn (142).

An HEXPE INVERSE:

"GILGAMESH – THE GOLDEN FORMULA" $A_j, B_j \in \mathbb{H}_R;\ B_j{}' \in \mathbb{H}_L;\ H \equiv h, h^{-1} \in \mathbb{X}_n.$

$$h \;=\; A_1 B_1{}' + A_2 B_2{}' + \cdots + A_n B_n{}' \;=\; \sum A_i B_i{}' \tag{A-59}$$

"Gilgamesh was two-thirds god and one-third man, Enkidu was two-thirds animal and one-third man, this is the story of them becoming hu-man together..."[18]

$$h^{-1} \;=\; \frac{h^*(hh^*)^{*R} + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}}{h^*(hh^*)^{*R}h + 2(h^{*L}(h^{*R}h^{*L})^{*R})^{*R}h} \tag{A-60}$$

$$= \frac{\sum \left(A_i^* A_k A_j^* + 2 \cdot A_j^* A_k A_i^*\right) B_i{}'^* B_j{}' B_k{}'^*}{\sum \left(A_i^* A_k A_j^* A_l + 2 \cdot A_j^* A_k A_i^* A_l\right) B_l B_k^* B_j B_i^*} \tag{A-61}$$

QUATERNION EXPANSION OF THE ADJOINT MATRIX

$$H^\dagger \;=\; \tfrac{1}{3} \cdot \text{MAN} + \tfrac{2}{3} \cdot \text{GOD} \tag{A-62}$$

$$\text{MAN} \;=\; H^*(HH^*)^{*R} \tag{A-63}$$

$$\text{GOD} \;=\; (H^{*L}(H^{*R}H^{*L})^{*R})^{*R} \tag{A-64}$$

QUATERNION EXPANSION OF THE MATRIX DETERMINANT

$$det(H) \;=\; \tfrac{1}{3} \cdot H^*(HH^*)^{*R}H + \tfrac{2}{3} \cdot (H^{*L}(H^{*R}H^{*L})^{*R})^{*R}H \tag{A-65}$$

HOW WE FOUND THIS SOLUTION: This formula is certainly not obvious in any sense. Originally, when we wrestled with this problem, it seemed intuitive that we should multiply by partial conjugates to achieve the reduction of the L-H-S of the linear problem to scalar; $h\hat{q} = \hat{C} \to h^{*R}h\hat{q} = h^{*R}\hat{C} \to (h^{*R}h)^{*L}h^{*R}h\hat{q} = (h^{*R}h)^{*L}h^{*R}\hat{C}$. This idea made alot of sense, since the first multiplication step reduces the L-H-S factor from two-hand quaternion to one-hand quaternion, which is left handed in this case, and we know how to invert the one-hand quaternion, so problem solved. We could start out with either factor, h^{*R} or h^{*L}, the latter giving us the solution path, $h\hat{q} = \hat{C} \to h^{*L}h\hat{q} = h^{*L}\hat{C} \to (h^{*L}h)^{*R}h^{*L}h\hat{q} = (h^{*L}h)^{*R}h^{*L}\hat{C}$, instead. But, this intuitive method only worked for "one term" and "two term" linear problems. When we came to the "three term" problem, we couldn't solve it this way. Yet, we knew the solution to the three term problem from our previous attacks on the problem using the method of guessing a factor with free parameters that could be later fixed to match the solution. The two cubes, $(h^{*R}h)^{*L}h^{*R}$ and $(h^{*L}h)^{*R}h^{*L}$, now formed the starting point in the search for solution. We had to start thinking about combining factors, but how to guess the right combinations? We then calculated the basis components of these hypercomplex numbers and compared them to our previously calculated inverse, h^{-1}, which was already known in basis component format, through a rather tedious but straighforward matrix algebra. This gave us our table of 360 cubes. Initially, a bug in our symbolic source code prevented us from seeing the right results for all these cubes. But, we were really focused on that pair of cubes that started out working anyway, and the rest of the table was just an interesting exercise. We saw that these two cubes could be easily combined to match the 15-dimensional vector part of the numerator for that inverse, h^{-1}, so the problem became a matter of finding suitable cubes to fit the scalar component, w_0. The double conjugated cube, $h^{*R}hh^{*L}$, fit the bill, and was selected to form that set of 3 cubes, that when conjugated into the four conjugate states, _ N R L , gave us the 12 final cubes necessary to write the adjoint down in terms of a quaternion expansion. So, we had found the solution to the arbitrary n-term linear problem. October 27, 2007, was our scheduled date to release the paper with our results, that gave us enough time to write up the paper, and the 27th was a cube, 3^3, in perfect sync with the cubic expansions.

However, on re-checking the unfinished paper the day before scheduled release, we noticed something strange about our cube table. Some of the sequences in the R-COLUMN were missing in the L-COLUMN. But, if we had the complete universe of all possible conjugated cubes, then every sequence found in the R-COLUMN must also appear somewhere in the L-COLUMN. So, either we had counted the number of cubes incorrectly, and there were more than 360 cubes, or there was a bug in our source code that produced errors in the column output. On thinking about this problem, we realised that by looking at the "corresponding cubes" we could decide the issue, and this led us to easily find the bug in our source code. Now, however, we paid more attention to that cube table. Although the solution we initally found did not depend on the table, and we even thought of just deleting the cube table from the paper to save time, the results of the table were important in giving us the confidence that all our other work was ok, since every output was a verification check of the source code working correctly. So, we couldn't ignore the table. Better to delay the paper. That's when we found "the Gilgamesh solution." Just 2 cubes, could accomplish the same task, where our previous solution required 12 cubes! But there was no logic to these two new cubes. There was no intuition, built up from familiarity with the struggle of tackling these linear problems, that would enable the mind to divine their construction aforehand. They just pop out like magic from a miscellaneous symbolic computation, with no analytical prepration in advance to warn the mathematical scientist that they might be lurking out there somewhere, and so he should seek them out. In fact, our focus was on the cubes that had the most isolated zeros 0s in their column entries, since that represented a perfect match to those parts of the adjoint. But, these two cubes didn't match any component of the adjoint at all. Yet, their differences off-set each other so perfectly, that we need only these two cubes, 1/3 of one and 2/3 the other, to write down the adjoint.

It is possible that more interesting things might be revealed in the future when this cube table has been explored further, and in greater depth. But, for the moment, this 2-cube solution is the reigning "king" of all the many quaternion expansions of the adjoint matrix and the determinant. An intriguing observation is that Khufu's pyramid is truncated to the height of a cube, $\mathrm{H}^3 = \mathrm{HHH} = \mathrm{VOLUME}$, and contains *several* obvious references to the theme of "inversion"–e.g. invert a cube to obtain six five sided pyramids, each base to a face and apex at center–Egyptian mythology encodes special reference to the cubesquare, in the Horus Eye $64 = 2^6$, and the large scale number 10^6 depicted by the Hieroglyph of a man kneeling and holding his right hand and left hand up to the heavens[19]; compare that the "inversion" of the four-dimensional 'right-hand + left-hand' transformation matrix $\mathrm{H} = \sum AB'$ involves the same motif of the square of the cube, $AAA.BBB = A^6$ when $B = A$, i.e. the sixth power of number is also involved in inversion. Do the pyramids and Egyptian mythology encode hints and clues to the mathematics of spacetime?

Miscellaneous Calculations.

Theorem: $(gh)^* = h^* g^* \quad \forall\, g, h, \in \mathbb{X}_n$.

Proof: Every hexpe number, $g, h \in X_n$, can be represented as the sum of R·L pair products, so, using the table of rules given on the first page;

$$
\begin{aligned}
g &= \sum A_j B'_j\,, \qquad h = \sum C_k D'_k\,, \qquad A_j, C_k \in \mathbb{H}_R;\ \ B'_j, D'_k \in \mathbb{H}_L \\
\therefore\quad (gh)^* &= \left(\left(\sum A_j B'_j\right)\cdot\left(\sum C_k D'_k\right)\right)^* = \left(\sum\sum A_j B'_j C_k D'_k\right)^* = \left(\sum\sum A_j C_k B'_j D'_k\right)^* \\
&= \sum\sum \left(A_j C_k B'_j D'_k\right)^* = \sum\sum \left(A_j C_k\right)^* \left(B'_j D'_k\right)^* = \sum\sum C^*_k A^*_j D'^*_k B'^*_j \qquad \text{(A-66)} \\
&= \sum\sum \left(C^*_k D'^*_k\right)\left(A^*_j B'^*_j\right) = \left(\sum C^*_k D'^*_k\right)\left(\sum A^*_j B'^*_j\right) = \left(\sum (C_k D'_k)^*\right)\left(\sum (A_j B'_j)^*\right) \\
&= \left(\sum C_k D'_k\right)^* \left(\sum A_j B'_j\right)^* = h^* g^* \qquad \text{Q.E.D.}
\end{aligned}
$$

[PJ2] P.M. Jack *Hexpentaquaternions: a two-hand quaternion algebra*, Jan 29, 2006.

[PJ3] P.M. Jack *Quatro-Quaternions and the matrix representations of octonions*, Jul 02, 2006.

[WRH1] W. R. Hamilton *Lectures on Quaternions*, 1853.

[YT1] Tian, Y *The equations* $ax - xb = c$ *,* $ax - \bar{x}b = c$ *, and* $\bar{x}ax = b$ *in quaternions.*, 2004. Southeast Asian Bulletin of Mathematics 28, 343-362.

[1] As early as 1853, in his "*Lectures on Quaternions,*" Hamilton writes "…for the SOLUTION OF EQUATIONS IN QUATERNIONS… very much remains still to be done towards the attainment of anything approaching to perfection in the establishment of *general methods for such solutions of equations* , and for QUATERNION ELIMINATION generally. **But so far as regards** EQUATIONS OF THE FIRST DEGREE **in quaternions, I have been for some years in possession of what appears to me to be such a general method of solution.**" [pg.522 sec.513.]. Hamilton then gives brief mention of this *general* first degree method in sec.514, illustrating his use of the $q = Sq + Vq$ decomposition to effect the solution, but does not go into depth on the method he believes he has found!

[2] By isomorphism, $\mathbb{H}_R \equiv \mathbb{H}_L$, we conclude, $(AB)^* = B^*A^* \implies (A'B')^* = B'^*A'^*$, etc… and together the rules in this table give us other things like, $(gh)^* = h^*g^*$, $(gh)' = h'g'$, $\forall\ g, h, \in \mathbb{X}_n$, once we accept the usual associative and distributive laws, which are also assumed to apply to the operators, * and $'$, i.e. $(g+h)^* = g^* + h^*$, and, $(g+h)' = g' + h'$.

[3] Index pattern is $((1+2)((1+2)))((1+2)) = (111+112+121+122+211+212+221+222)$, on A^*AA^* only.

[4] For proof $(gh)^* = h^*g^*$, See MISCELLANOUS CALCULATIONS at the end of the Appendix.

[5] Note also; $A_1^*A_2 + (A_1^*A_2)^* = A_1A_2^* + (A_1A_2^*)^*$, etc...i.e. $S(A_1A_2^*) = S(A_2A_1^*) = S(A_1^*A_2) = S(A_2^*A_1)$, i.e. these expressions which resolve to scalars can swap the conjugates througout, as well as permute the factors. Also hands, $S(A_1^*A_2) = S(A_1'^*A_2')$.

[6] The norms can all be replaced by minus squares, e.g. $|X|^2 = -(X - X_0)^2$ or $|X|^2 = -(VX)^2$, and so removed from the formulas entirely. But, they help remind us that these are scalar factors, and, as such, are useful, particularly in writing the denominator which evaluates to a scalar.

[7] This can most easily be seen by re-writing, $|G_2|^2 = G_2^*G_2 = 2|B_1|^2|B_3|^2 - (B_2^*B_1B_3^*B_2B_1^*B_3 + (B_2^*B_1B_3^*B_2B_1^*B_3)^*)/|B_2|^2$

[8] Index pattern arrangement is $((1{+}2{+}3)((1{+}2{+}3)))((1{+}2{+}3)) = 111+112+113+121+122+123+131+132+133+211+212+213+221+222+223+231+232+233+311+312+313+321+322+323+331+332+333$, on A^*AA^*.

[9] Division from left and from right are notated, $A\backslash B \equiv A^{-1}B$, and $B/A \equiv BA^{-1}$. By convention, the \ and / take precedence over the multiplication $\cdot$ in this paper, so that, $A\backslash B \cdot C\backslash D \equiv (A\backslash B)\cdot(C\backslash D)$. Also, note the hand transformation rules, $(A/B)' = B'\backslash A'$ and $(A\backslash B)' = B'/A'$; conjugation has the same effect of reversing the divisor slash.

[10] To compare this result to that obtained using (54), note that, for any c, b quaternions, $[c, b] = [b^*, c]$, i.e. $cb - bc = b^*c - cb^*$, therfore, $-|a|^2cb = -|a|^2bc - |a|^2b^*c + |a|^2cb^*$; the five terms in the numerator then reduce to four.

[11] The term "distinguished bases" was introduced by Prof. Edwin Clark to describe my particular $\mathbb{H}\otimes\mathbb{H}'$ representation, in contrast to Floretions $\mathbb{H}\otimes\mathbb{H}$, on sci.math.research:
`http://www.mathkb.com/Uwe/Forum.aspx/research/2365/Hexpentaquaternions-a-two-hand-quaternion-algebra`.

[12] Strictly speaking, this space reflection is acompanied by a time reversal, since the scalar component flips sign also under the i_Li_R operation.

[13] Although in our notation, $A + A^*$ is twice the scalar part of A, we really reference the fact that the expression is *real valued*, so commutes, rather than emphasise it's component characteristic; we never extract the vector part, for example. However, the notation $S(PQRS)$ is convenient shorthand for discussing the invariance of scalars under cyclic permutation of factors. And in our Appendix, we reference components only to verify and establish the general rules and formulas; after these results are accepted, they become the starting points for reckoning, and we need not refer to the components thereafter.

[14] *I regard it as an inelegance, or imperfection, in quaternions, or rather in the state to which it has been hitherto unfolded, whenever it becomes or seems to become necessary to have recourse to x, y, z, etc.* William Rowan Hamilton (ed. Quoted in a letter from Tait to Cayley.) see "Quotes:" section at `http://en.wikipedia.org/wiki/Quaternion` . This famous quote also appears in the Preface of Tait's "An Elementary Treatise on Quaternions". The original is from Hamilton's 1853 "Lectures on Quaternions", page 522, sec:513, where the exact wording is, "**I regard it, however, as an inelegance and imperfection in this calculus, or rather in the state to which it has hitherto been unfolded, whenever it becomes, or** ***seems*** **to become, necessary to have recource, in any such way as this, to the resources of ordinary algebra, for the** SOLUTION OF EQUATIONS IN QUATERNIONS." See Hamilton's Lectures online at:
`http://cdl.library.cornell.edu/cgi-bin/cul.math/docviewer?did=05230001&seq=662&frames=0&view=50`

[15] Here we use compressed notation
(HR·H)L·HR for $(h^{*R}h)^{*L}h^{*R}$, etc.. an intermediate to RPN notation HRHXLHRX which we also use in this paper.

[16] i.e. for first powers, $\sim h$, the max irreducible conjugation is 1, for squares, $\sim hh$, the max irreducible conjugation is 3, and for cubes, $\sim hhh$, the max irreducible conjugation is 5.

[17] Reverse Polish Notation

[18] Adapted from the Ancient Akkadian poem, called "The Epic of Gilgamesh," about a 3rd millenium B.C. king of Uruk.

[19] In Revelation $10^{:6}$ and Daniel ^{12}: 7 the gesture is used in reference to the measure of time; $12/7 = 1.\mathbf{714285,7}\ldots$ pattern of "doubling" breaks, at the 6th decimal place, i.e. 10^{-6}; what would be $2\times 28 = 56$, is interrupted by the 7 following the 5, instead of 6 there. See also Psalms 90×4 and 2 Peter 3×8 for unit of time a cube 10^3 and cubesquare $10^3\times 10^3 = 10^6$.